Some Recent Results and Open Questions on Two Phase Free Boundary Problems

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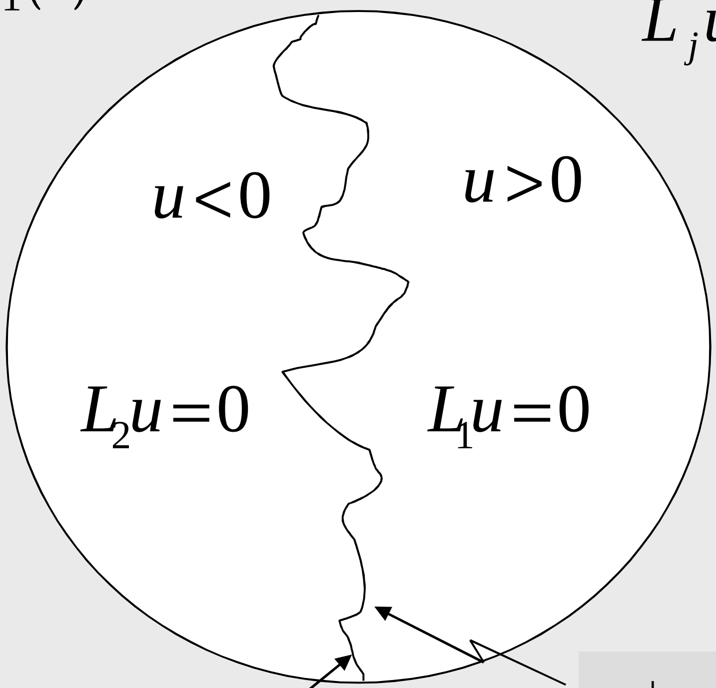
Elliptic f.b. problems

$B_1(0) \subset R^n$



$$L_j u = \mathrm{Tr}(A^j(x) D^2 u) + b^j(x) \cdot \nabla u$$

or

$$L_j u = \Phi_j\left(D^2 u, Du, x\right)$$

or

$$L_j u = \mathrm{div}(A^j(x) \nabla u)$$

$$u_\nu^+ = G\left(u_\nu^-, \nu, x, \ldots\right)$$

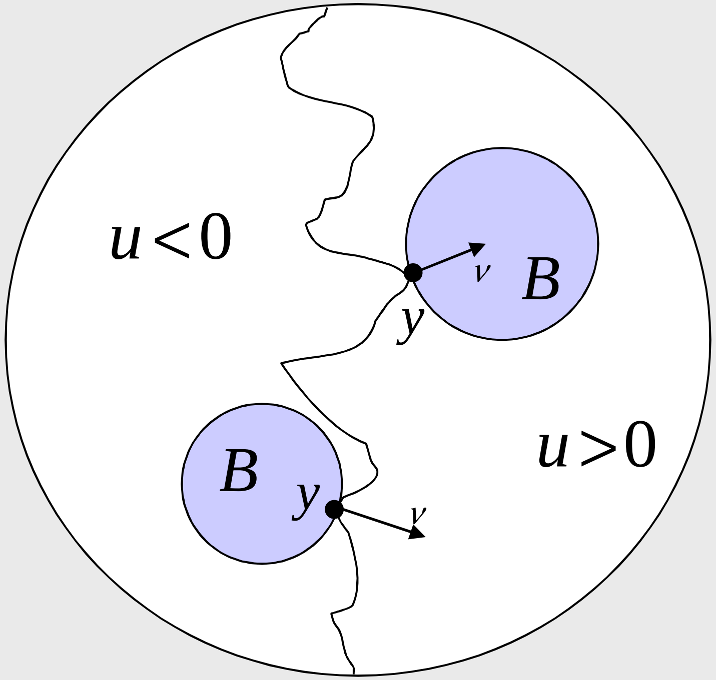
$F(u) = \partial\{u>0\}$

free boundary

Viscosity solution:

u continuous and whenever at *y* on *F(u)* there exists a touching ball *B*,

$$u(x) = \alpha < x-y, \nu >^{+} - \beta < x-y, \nu >^{-} + o(|x-y|)$$



$B \subset \{u > 0\} \Rightarrow \alpha \leq G(\beta, \nu, x)$

(*y regular from the right,* supersolution condition)

$B \subset \{u < 0\} \Rightarrow \alpha \geq G(\beta, \nu, x)$

(*y regular form the left,* subsolution condition)

*Hypotheses on **G**:*

$z \mapsto G(z,\nu,x)$ strictly increasing in $(0,\infty)$

$z \mapsto z^{-N} G(z,\nu,x)$ decreasing in $(0,\infty)$, N large

$(\nu,x) \mapsto G(z,\nu,x)$ Lipschitz

Examples

$$\int_{B_1} \left\{ a_{ij}(x) u_{x_i} u_{x_j} + q(x) \chi_{u>0} \right\} dx$$

$$u \in H_0^1(B_1) + g,\ a_{ij}, q \geq c > 0 \text{ smooth.}$$

$$\left(u_{\nu*}^{+}\right)^2 - \left(u_{\nu*}^{-}\right)^2 = q(x)$$

$$\Delta u_\varepsilon + b(x) \cdot \nabla u_\varepsilon = \beta_\varepsilon(u)$$

$$\text{support}\beta \subset [0,1], \int \beta = 1, \beta_\varepsilon(s) = \varepsilon^{-1} \beta(s/\varepsilon)$$

$$(u_\nu^{+})^2 = 2$$

Existence

Regularity

of the solution

of the free boundary

weak results

(measure theor.)

strong results

$Lip \Rightarrow$ (e.g.) $C^{1,\alpha}$

$"\,flat\," \Rightarrow Lip$ or $C^{1,\alpha}$

L. Caffarelli

Regularity

$$L_1 = L_2 = \Delta$$

Rev. Ibero Am. 1987: $Lip \Rightarrow C^{1,\alpha}$

C.P.A.M. 1988: $"Flat" \Rightarrow Lip$

Existence

$$L_1 = L_2 = div\left(A(x)\nabla\right)$$

$$G(0,\nu,x) \geq c > 0$$

↑

Hoelder continuous

Ann. SNS 1988 ⟶

There exists a viscosity *solution u of the Dirichlet problem* (continuous data). *Moreover u is Lipschitz, $\Omega^{+}(u)$ is a set of finite perimeter and*

$$0 < \alpha_1 \leq \frac{u^{+}(x)}{d(x, F(u))} \leq \alpha_2$$

Perron method

$$u(x) = \inf_{v \in S} v(x)$$

S class of continuous supersolutions v in Ω Lipschitz, $v \geq \varphi$ on $\partial\Omega$ such that $F(v)$ is regular from the left.

$$L_1 = L_2 = F\left(D^2 u\right),$$
$$F \text{ concave}$$

P.Y. Wang (*J. of Geom. An. 2000)*

open for:

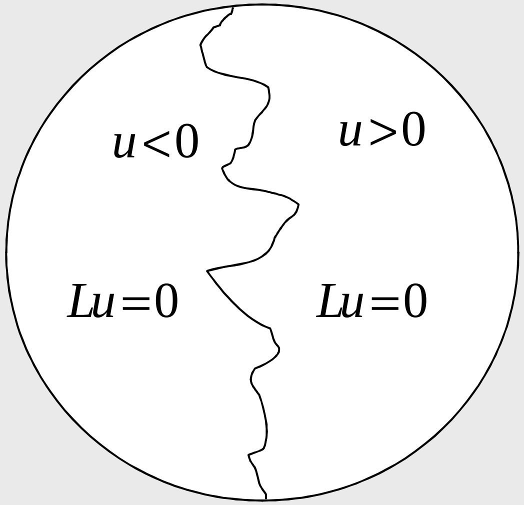
$$F = F\left(D^2 u, Du\right) \text{ non concave}$$

and

$$Lu = \mathrm{Tr}(A(x)D^2 u) + b(x)\cdot\nabla u$$

Key point: **monotonicity formula of Alt-Caffarelli-Friedman** (*TAMS 1983)*

$$u = u^+ - u^-, div A\nabla u^\pm \geq 0,\ u^\pm(0) = 0$$



$$\Downarrow \quad r \leq 1/2$$

$$\Phi(r) = r^{-4} e^{-cr^\alpha} \int_{B_r} \frac{|\nabla u^+|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla u^-|^2}{|x|^{n-2}} dx \leq c_n |u|_\infty^4$$

is increasing and

$$\left(\partial_\nu u^+(0)\right)^2 \left(\partial_\nu u^-(0)\right)^2 \leq \Phi(1/2)$$

$$Lu = \mathrm{Tr}(A(x) D^2 u) \ ? \quad \Longleftarrow \quad \frac{u^\pm(x)}{d(x, F(u))} \approx |\nabla u^\pm(x)|$$

Regularity $Lip \Rightarrow C^{1,\alpha}$

$$L_j = a_{ij} D_{ij}$$ **Feldman**, Diff. Int. Eq. 1997

$$L_1 = L_2 = F\left(D^2 u\right),$$ **P.Y. Wang**, C. P.A.M., 2000

F concave

also *flat* $\Rightarrow$ *Lip*

$$L_j = F_j\left(D^2 u, Du\right)$$ **Feldman**, Indiana, 2001

$$L_j u = \mathrm{Tr}(A^j(x) D^2 u)$$ **Cerutti, Ferrari, S.** Arch. Rat. Mec. 04

$$L_j u = \Phi_j\left(D^2 u, Du, x\right)$$ **F. Ferrari,** Am. Jour. of Math. 05

$$L_j u = \mathrm{Tr}(A^j(x) D^2 u) + b^j(x) \cdot \nabla u$$ **Ferrari, S.** Adv in Math 2007

Theorem 1 *(Lipschitz implies smoothness)*

Let u be a viscosity *solution of the f.b.p. Assume:*

$i)\ A^j \in C^{0,\alpha}(B_1),\ b \in L^\infty(B_1).$

$ii)\ F(u) = \{x_n = f(x')\}$ *with f Lipschitz*

Then f is $C^{1,\gamma}$ *graph in* $B'_{1/2}$.

Corollary: Conclusion of Theorem 1 holds if:

$$L_1 u = L_2 u = div\left(A(x,u)\nabla u\right)$$

with A Lipschitz (w.r.t. x, u)

open for *A(x,u) Hoelder*

Theorem 2 *(flatness and nondeg. of (u+) implies smoothness)*

Let u be a viscosity *solution of the f.b.p. Assume:*

i) $A^j \in C^{0,\alpha}(B_1)$, $b \in L^\infty(B_1)$.

ii) $0 < \alpha_1 \leq \dfrac{u^+(x)}{d(x, F(u))} \leq \alpha_2$

iii) $G(0, \nu, x) \geq c > 0$

There exist $0 < \bar{\theta} < \pi/2$ *and* $\bar{\varepsilon} > 0$ *s.t. if for* $0 < \varepsilon < \bar{\varepsilon}$ $F(u)$ *is contained in an* ε*-neighborhood of a graph of a Lipschitz function* $x_n = g(x')$ *with*

$$\mathrm{Lip}(g) \leq \tan\left(\frac{\pi}{2} - \bar{\theta}\right),$$

then $F(u)$ *is a* $C^{1,\gamma}$ *graph in* $B_{1/2}$.

Theorem 2

$$L_j u = \Phi_j\left(D^2 u, Du, x\right)$$

R. Argiolas, F. Ferrari,

(Interphase and F.B. 2009)

$$L_1 = L_2 = div\left(A\left(x\right)\nabla\right)$$

$$G\left(0,\nu,x\right)\geq c>0$$

Theorem 3 *(Smoothness of minimal solutions)*

Let u be the minimal viscosity *solution of the Dirichlet problem.*
Assume $A = A\left(x\right)$ *is Lipschitz.*
If $x_0 \in \partial_{red}\Omega^+\left(u\right)$ *then* $F\left(u\right)$ *is a* $C^{1,\alpha}$ *surface in a neighborhood of* x_0

➡ **open for** *A(x) Hoelder*

Main point:

Let L be a second order uniformly elliptic operator,
u continuous and Lu = 0 on its support.
Let g > 0 be a smooth function and define

$$v_g(x) = \sup_{B_{g(x)}(x)} u$$

Under which condition on g is v_g L-subharmonic on its support?

L. Caffarelli (1987):

$$Lu = \Delta u = 0 \ on \ \{u \neq 0\}$$

v_g **is subharmonic in its support if**

$\varphi \in C^2, \ 1 \leq \varphi \leq 2$ and $$\varphi \Delta \varphi \geq C(n) |\nabla \varphi|^2$$

$$Lu = \mathrm{Tr}(A(x) D^2 u) + b(x) \cdot \nabla u$$

$$Lu = \mathrm{div}(A(x) \nabla u)$$

?

Cerutti, Ferrari, S. (2004, 2007)

$$Lu = \mathrm{Tr}(A(x)D^2u) + b(x)\cdot\nabla u$$

v_g **is L-subharmonic in its support, if**

$$g \in C^2,\ 1 \le g \le 2,\ |\nabla \mathrm{g}| \le \mu\left(n, \Lambda, |b|_\infty\right)$$

$$Lg \ge C(n,\Lambda)\left\{ \frac{|\nabla g|^2 + \omega^2}{g} + |b|_\infty \right\} \qquad (*)$$

$\omega = \omega(\max g/\Lambda)$ **(Modulus of continuity of the coefficients)**

⇒ $L = div(A(x)\nabla)$ *requires Lipschitz coefficients* (without the term $|b|_\infty$)

Open:

- *Existence in the cases*

$$Lu = \Phi\left(D^2u, Du\right)$$ Φ **non concave**

$$Lu = a_{ik}\left(x\right)u_{x_i x_k} + b_k\left(x\right)u_{x_k}$$ **Monotonicity formula**

- ***Regularity under minimal smoothness on the coefficients***

$$Lu = \partial_{x_k}\left(a_{ik}\left(x\right)u_{x_i}\right)$$ **with Hoelder (or Dini!) coefficients**

L-**subharmonicity of** $v_\varphi\left(x\right) = \sup_{B_{\varphi(x)}(x)} u$

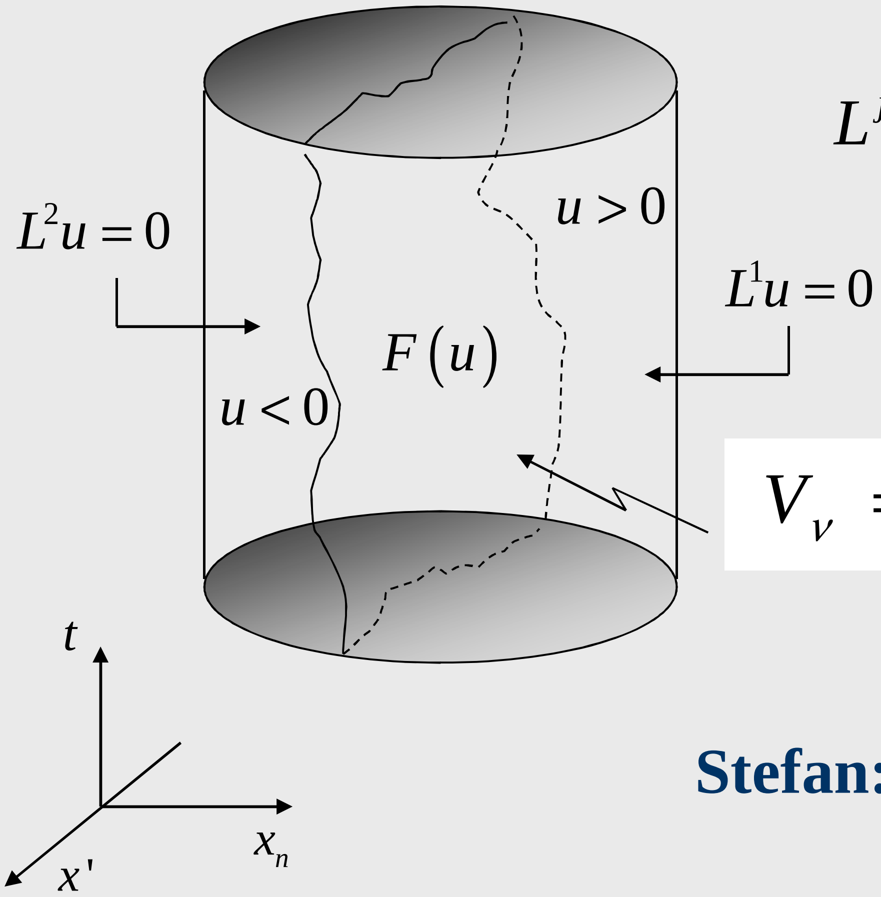
Evolution f.b. problems

$$C_1 = B_1 \times (-1,1)$$

$$L^j u = F\left(D^2 u\right) - u_t,\ F \ \textit{concave}$$

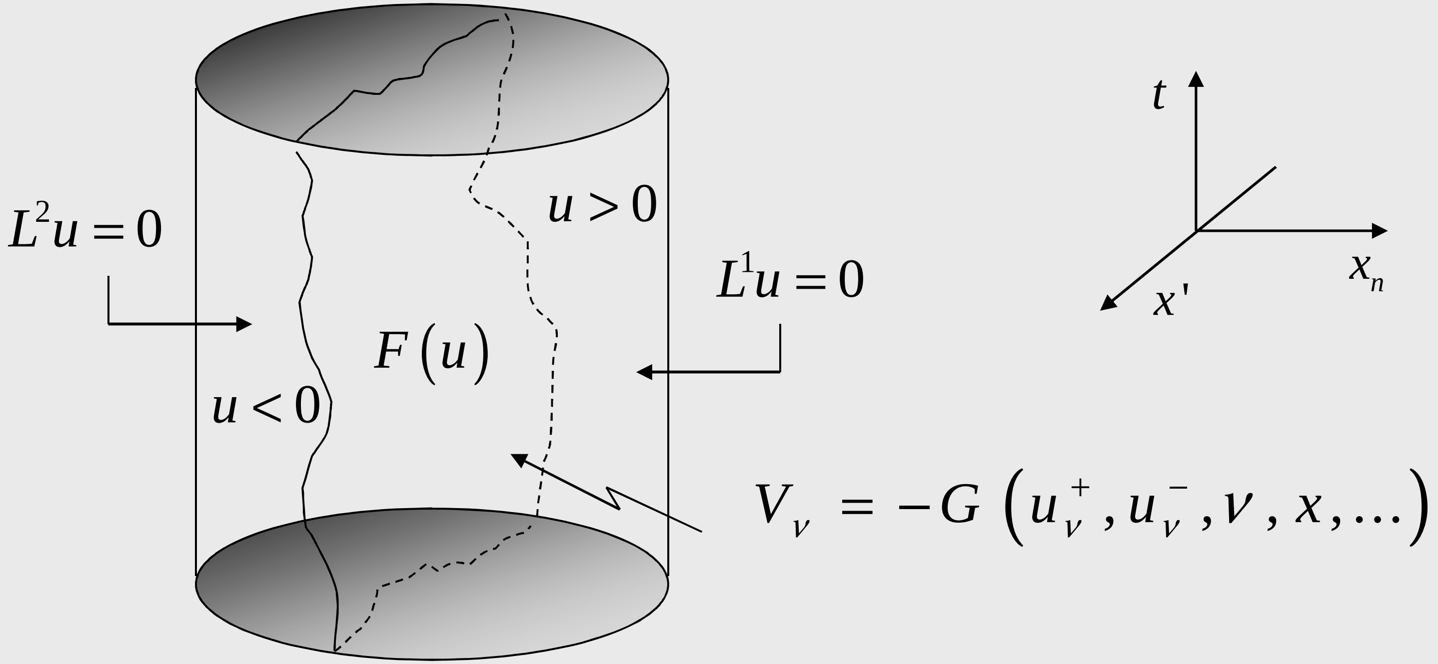
$$L^j u = a_{ik}^j (x,t) u_{x_i x_k} + b_k^j (x,t) u_{x_k} - u_t$$

$$L^j u = \partial_{x_k} (a_{ik}^j (x) u_{x_i}) - u_t$$



$$V_\nu = -G\left(u_\nu^+, u_\nu^-, \nu, x, \ldots\right)$$

Stefan: $G\left(u_\nu^+, u_\nu^-\right) = u_\nu^+ - u_\nu^-$


$L^2 u = 0$
$u > 0$
$L^1 u = 0$
$F(u)$
$u < 0$
$V_\nu = -G\left(u_\nu^+, u_\nu^-, \nu, x, \ldots\right)$
t
x_n
x'

Viscosity solutions

A continuous function *v is a viscosity subsolution (super)* in a domain *D* if for every sub cylinder *Q* and every classical super (sub) solution *w*:

$$v \le w \;\; (v \le w) \;\; on \; \partial_p Q$$

$$\Rightarrow$$

$$v \le w \;\; (v \le w) \;\; in \; Q$$

Existence known only for the heat equation with Stefan f.b. condition

Regularity $L^j u = a^j \Delta - u_t$

Athanasopoulos, Caffarelli, S.

Optimal regularity of the solution (Ann. of Math 1996)

$F(u)$ *Lip* in space and time $\Rightarrow$ u space-time *Lipschitz*

Regularity of the free boundary (Acta Math. 1996)

$F(u)$ *Lip* in space and time $\not\Rightarrow$ further regularity of $F(u)$

$F(u)$ *Lip* in space and time + nondegeneracy

$$\Rightarrow$$

$F(u)$ is C^1 and the $t-$sections are Liapunov-Dini domains, viscosity solutions are classical solutions

$Flatness \Rightarrow smoothness$ *(C.P.A.M 1998)*

- *Wayting time regularization phenomenon*
- *Istantaneous, local in time, regularization from "flat data"*
- *Stability of travelling waves*

Stefan: $G\left(u_\nu^+, u_\nu^-\right) = u_\nu^+ - u_\nu^-$

Koch, H. *(C.P.D.E., 1998)*

$F(u)$ is C^1 and the $t-$sections are Liapunov-Dini domains, the free boundary is C^∞.

Pruess, J, Saal, J., Simonett G *(Math. Annalen, 2007)*

$F_t(u)$ is *a graph, the initial data and the initial free boundary are* $C^{1,\alpha}$, + *non* degeneracy *and small oscillation of the normals of initial data,*

then, there exist a unique local (in time) solution of the Stefan problem and the free boundary is analytic in space and time.

$$Lu = a_{ik}(x,t)u_{x_i x_k} + b_k(x,t)u_{x_k} - u_t$$

?

$$Lu = \partial_{x_i}(a_{ik}(x)u_{x_k}) - u_t$$

Main tools and difficulties

- **control of *t-derivative* in terms of *x-gradient***

$$|u_t| \leq c\,|\nabla u|$$

➡ Space-time cone of monotonicity

Derivatives do not satisfies any equation

- **Results from potential theory**

Fatou theorems (for derivatives), mutual absolute continuity between caloric and surface measure

R. Argiolas R, A. Grimaldi,

Forum Math., 2008

Math. Nach. 2009

development of potential theory for non divergence equations

- **double homogeneity**

$$Lu = a_{ik}(x,t)u_{x_i x_k} + b_k(x,t)u_{x_k} - u_t$$

invariant under **parabolic** *rescaling*

$$V_\nu = -G\left(u_\nu^+, u_\nu^-, \nu, x, \ldots\right)$$

invariant under **hyperbolic** *rescaling*

➡ (miraculous !) choice of an adapted intermediate scaling

Ferrari, S.

Theorem *(Optimal regularity of the solution, C.P.D.E, 2010)*

u viscosity *solution in* $B_1 \times (-1,1)$. *Assume that* $F(u)$ *is given locally by* $x_n = f(x',t)$ *with* f *Lipschitz.*

If the Lipschitz constant in space *is small*

then: *in* $B_{1/2} \times (-1/2, 1/2)$

a) *u is Lipschitz*,

b) *$F(u)$ is locally C^1 in space, uniformly in time.*

Open:

$\Rightarrow$ $Lip + \text{nondegeneracy} \Rightarrow C^{1,Dini}$

$$L^1 u = L^2 u = a_{ik}(x,t) u_{x_i x_k} + b_k(x,t) u_{x_k} - u_t$$

$\Rightarrow$ $L^1 u = L^2 u = F(D^2 u) - u_t$

Milakis, *Indiana, 2005*

F *concave*

- $a_{ik}(x,t) = a_{ik}(x)$

⇨ a) u_{x_j}, u_t *have* non tangential limits *on* $F(u)$ *a.e. w.r. to surface measure*

⇨ b) *If* $(0,0)$ *is a differentiability point of* $F(u)$, *near* $(0,0)$

$$u(x,t) = \left(\alpha^+ x_n + \beta^+ t\right)^+ - \left(\alpha^- x_n + \beta^- t\right)^- + o\left(d_{x,t}\right)$$

with $\alpha^+ > 0, \alpha^- \geq 0$ *and*

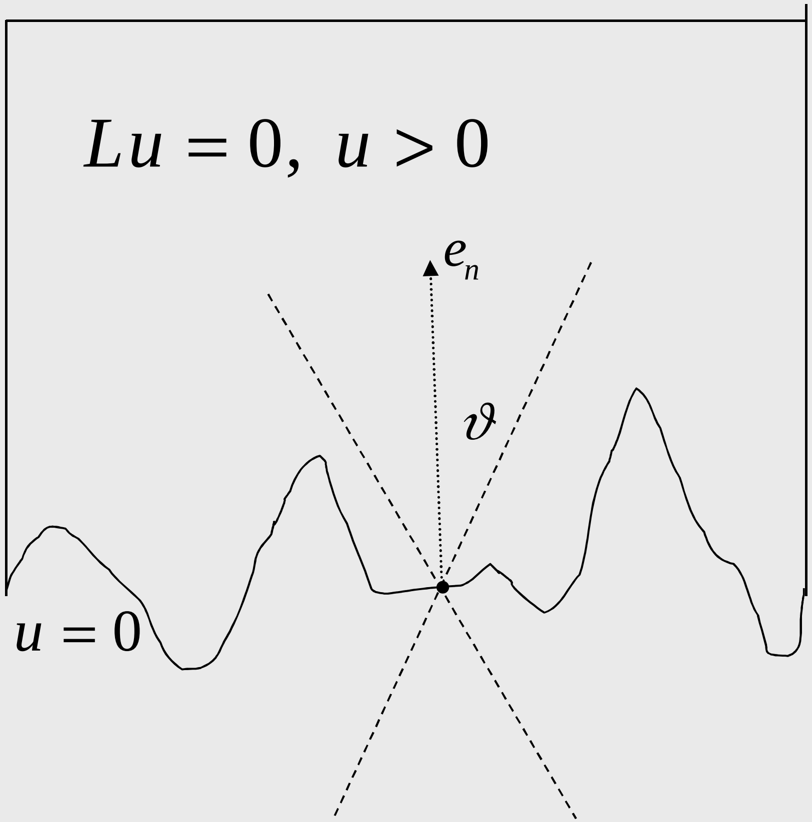
$$\beta^+ = \alpha^+ G\left(\alpha^+, \alpha^-\right)$$

⇨ c) $F(u)$ *is* C^1 *in space.*

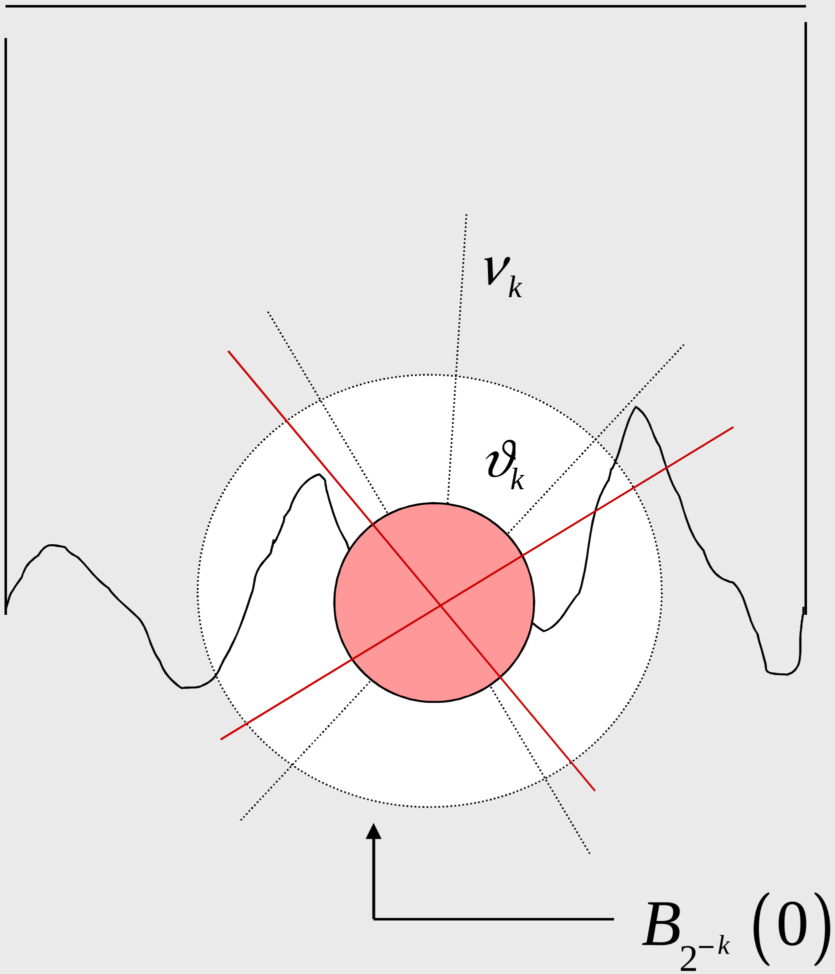
$Lip \Rightarrow C^{1,\alpha}$

Underlying idea: *geometric characterization of $C^{1,\alpha}$ domains*

$F(u)$ *Lipschitz* $\Rightarrow \exists$ a monotonicity cone $\Gamma_0(e_n,\theta)$ such that *if* $\tau \in \Gamma_0(e_n,\theta)$ *then* $D_\tau u \geq 0$



$in\ B_{2^{-k}}(0)$ *monotonicity cones* $\Gamma_k(\nu_k, \vartheta_k)$



$$\sigma \in \Gamma_k(\nu_k, \vartheta_k) \Rightarrow D_\sigma u \geq 0$$

$$\frac{\pi}{2} - \vartheta_{k+1} \leq \delta\left(\frac{\pi}{2} - \vartheta_k\right)$$

with $\delta < 1$

$$\alpha = -\log_2 \delta$$

$Lip \Rightarrow C^{1,\alpha}$

Step 1. To show that the level sets of *u* are uniformly Lipschitz graph

Step 2. To improve the Lipschitz constant of the level sets of the solution *u* away from *F(u)*

Step 3. To carry the interior gain to *F(u)*

Step 4. To rescale and iterate steps 2,3.

Strategy for T1

Step 1. To show that u^+ is (strictly) ε-monotone along the direction in a cone $\Gamma\left(e_n, \vartheta^*\right)$ with $\theta^* < \theta$ in a neighborhood of $F(u)$.

Fully monotone outside a $R\varepsilon-$neighborhood of $F(u)$.

$$u^+(x+\varepsilon'\tau) - u^+(x) \geq \varepsilon\lambda u^+(x)$$

(Perturbation from the constant coefficient case,
use of maximum and Harnack principles.
Same for div and non-div, hoelder continuous c.)

Step 2.

Dicothomy:

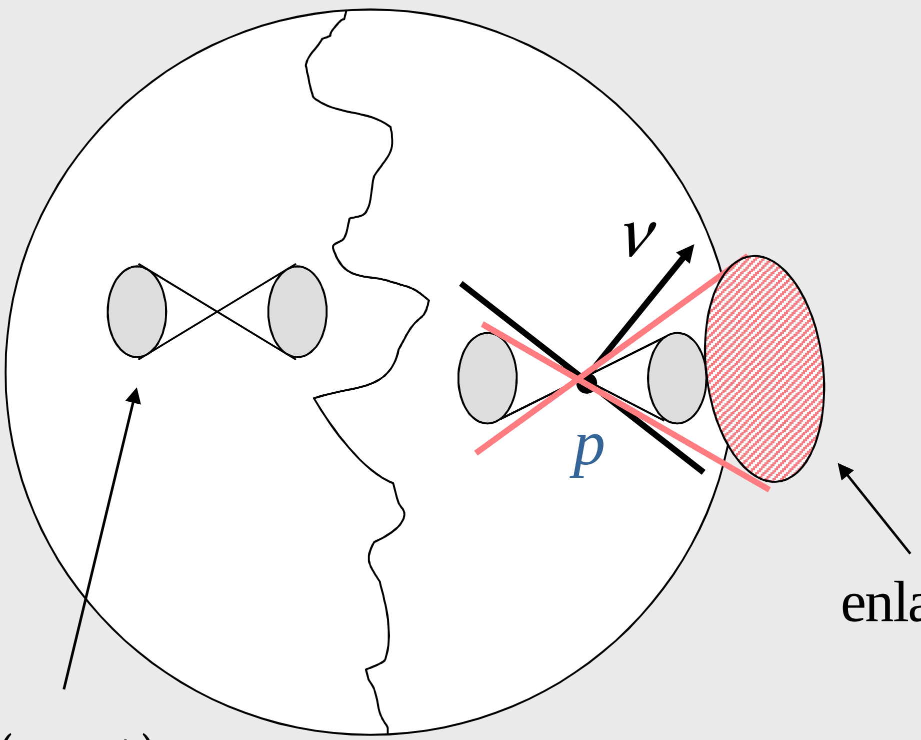
a) $u^{-} \geq C\varepsilon^{1/2} \max u \Rightarrow u$ strictly ε^{p}-monotone

b) $u^{-} \leq C\varepsilon^{1/2} \max u \Rightarrow u^{-}$ "negligeable"

(Perturbation from the constant coefficient case,
use of maximum and Harnack principles.
Monotonicity formula for div, barrier argument for non-div
both hoelder continuous c.)

⟹ a)

Step 3. *Enlargement of the cone far from F(u)*

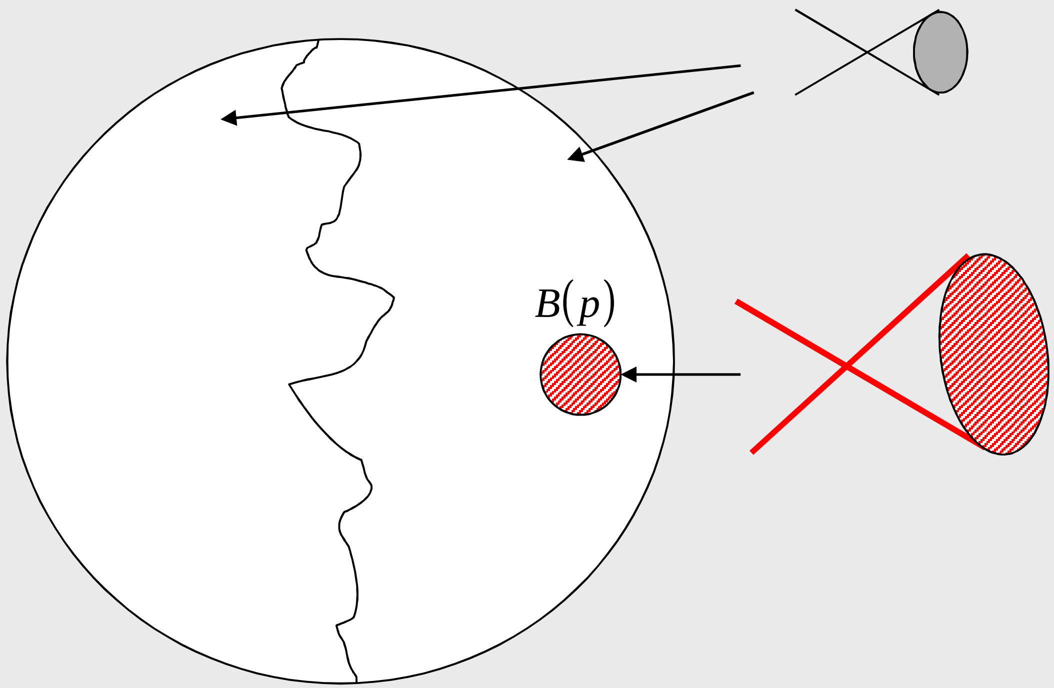


$$\nu = \nabla u(p)$$

enlarged cone

$\Gamma(e_n, \vartheta^*)$

Harnack inequality transfers the gain to *B(p)*

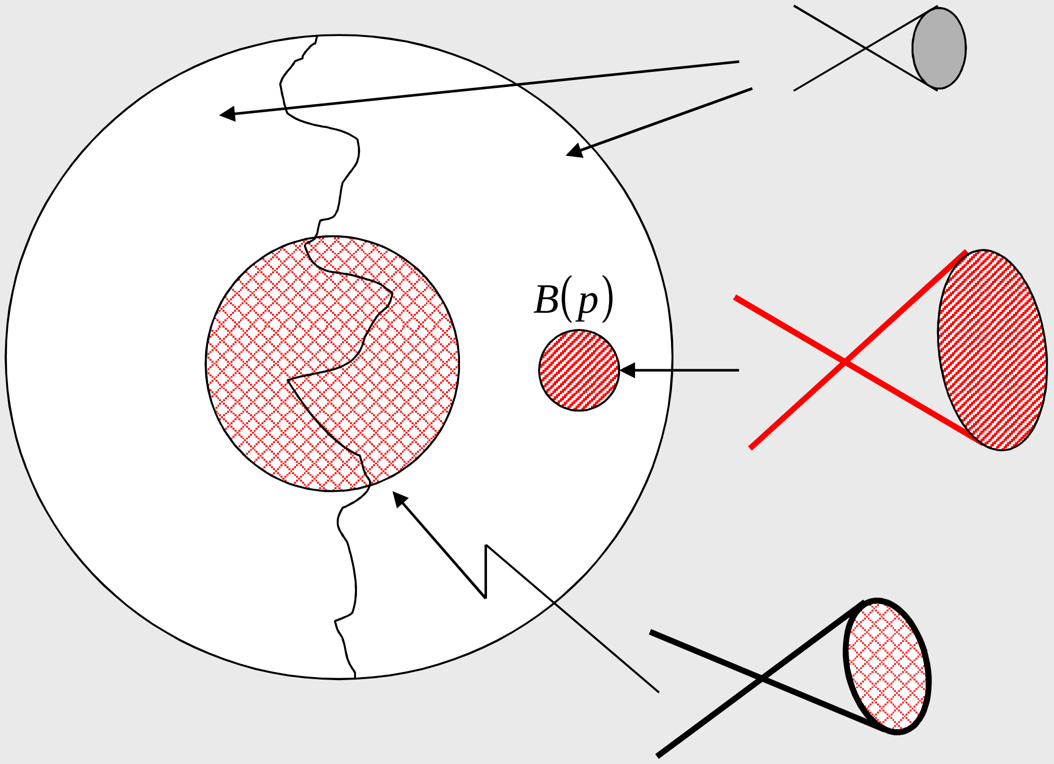


(Perturbation from the constant coefficient case,

use of maximum and Harnack principles.

Same for div and non-div, both hoelder continuous c.)

Step 4. Carry the interior gain in ε-monotonicity to $F(u)$ in $B_{1/2}$



main question

Step 5. Procedure for lowering ε to $\varepsilon/4$ in $B_{1/2}$.

(Not possible to lower ε to reach 0)

⇨ **main question**

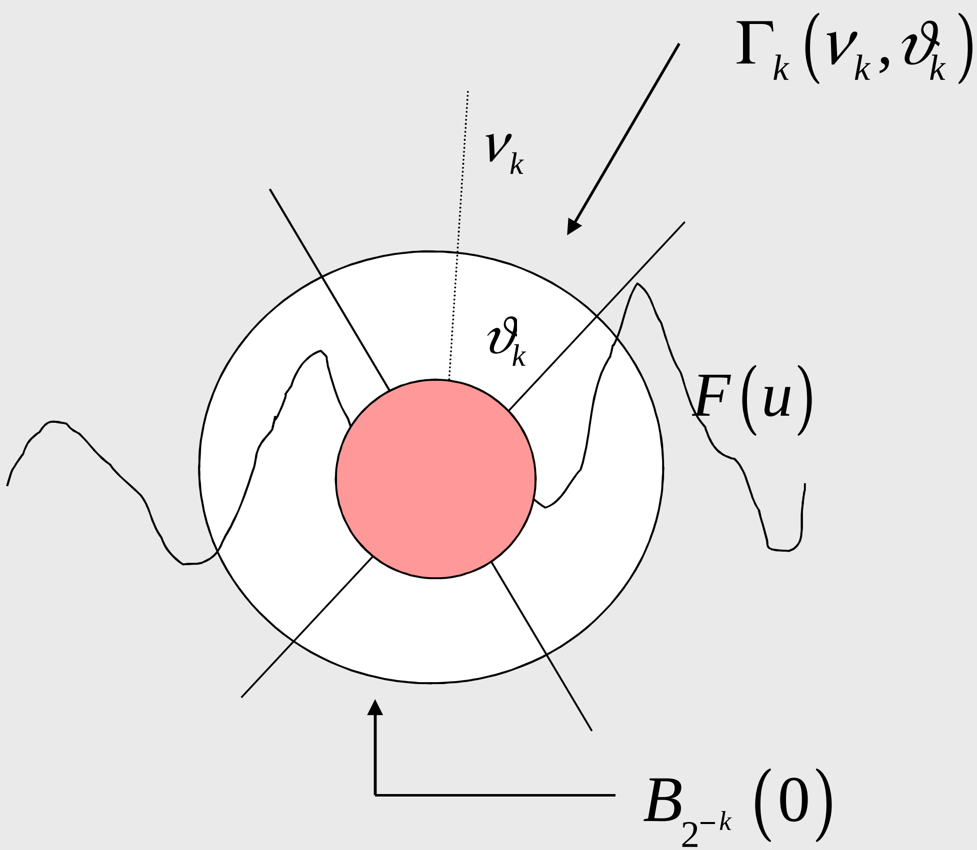
Step 6. Iteration of the above steps to construct a sequence of cones $\Gamma_k = \Gamma(\theta_k, \nu_k)$

$$\delta_k = \left(\frac{\pi}{2} - \vartheta_k\right)$$

$$\delta_{k+1} \le \rho' \delta_k + c_0 \varepsilon_k \quad (\rho' < 1)$$
$$|\nu_{k+1} - \nu_k| \le c\delta_k$$

such that u is $\varepsilon_k = 4^{-k}\varepsilon$ *monotone* in $B_{2^{-k}}$ along Γ_k.

ring $B_{2^{-k}} \setminus B_{2^{-k-1}}$, centered at a point on F(u)



$\delta_k \leq \rho^k \delta_0$

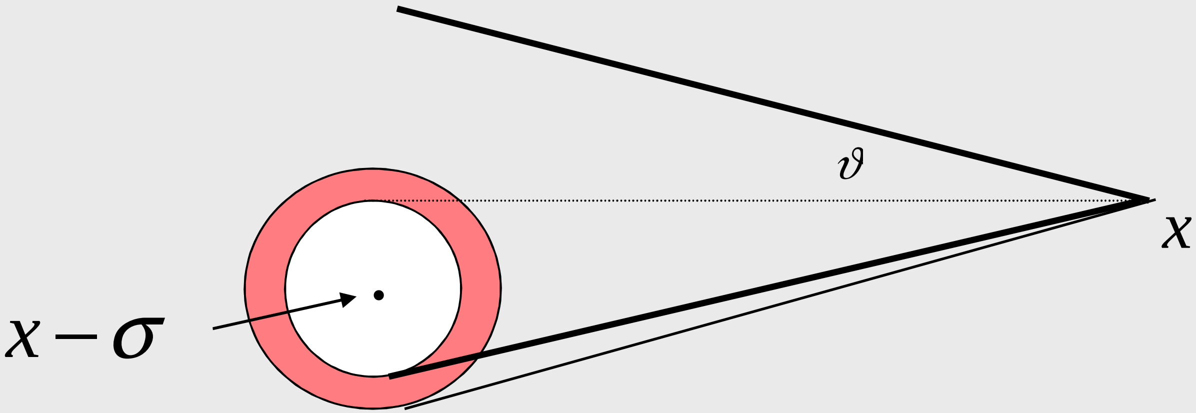
$\alpha \approx -\log_2 \delta$

Back to step 4.

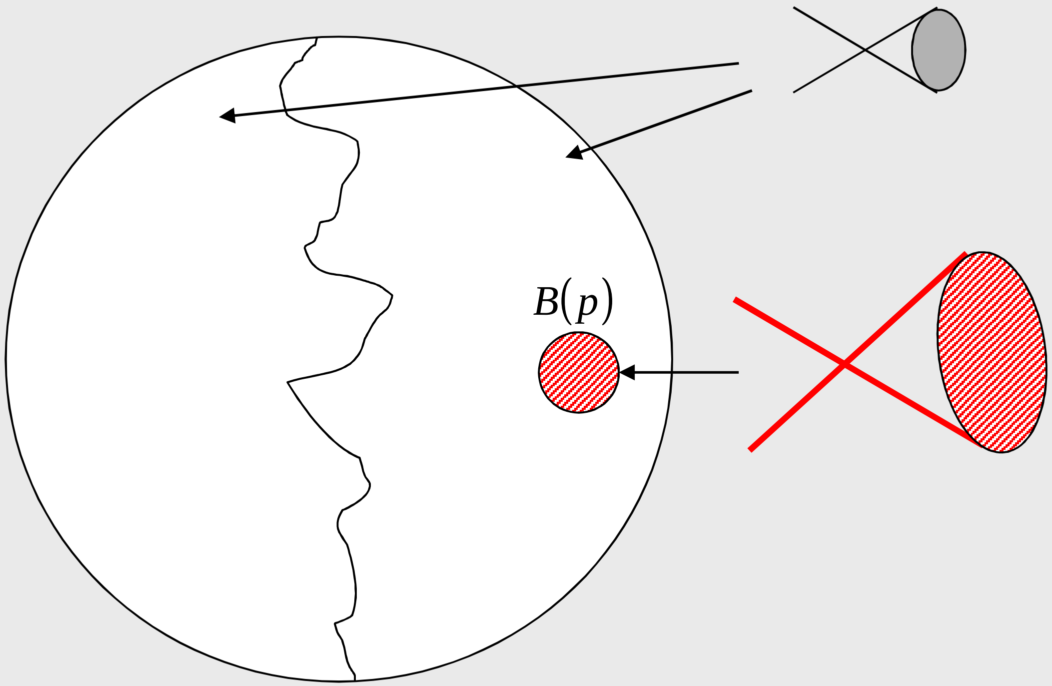
Enlargement of the cone up to F(u), in half ball

$$D_\sigma u \geq 0, \ \ \forall \sigma \in \Gamma(e_n, \vartheta)$$

$$\sigma \in \Gamma(e_n, \vartheta/2), \quad \varepsilon = |\sigma| \sin(\vartheta/2)$$



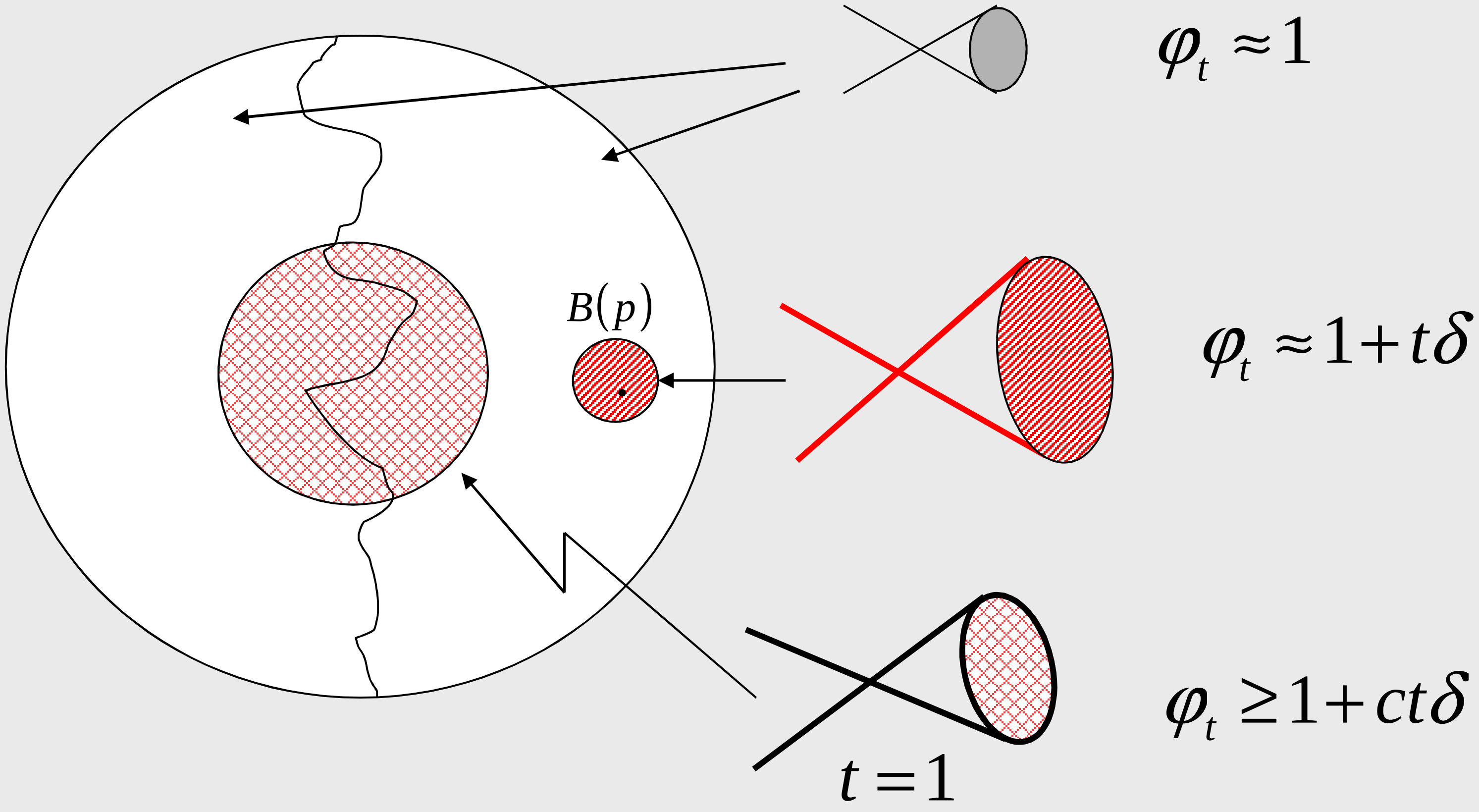
$$v_\varepsilon(x) = \sup_{B_\varepsilon(x)} u(y-\sigma) \leq u(x)$$



$$v_{\varepsilon\varphi_t}(x) = \sup_{B_{\varepsilon\varphi_t(x)}(x)} u(y-\sigma) \qquad 0 \le t \le 1$$

family of variable radii

Choice of φ_t such that



$$V_{\varepsilon\varphi_t}(x) = v_{\varepsilon\varphi_t}(x) + small\ correction\ term$$

is a family of subsolutions and satisfies the hypotheses of the following

Theorem (*continuity method*).

$Let\ v_t, 0 \le t \le 1,\ be\ a\ family\ of\ subsolutions,$

$continuous\ in\ D \times [0,1]\ and\ u\ be\ a\ solution\ of\ the\ f.b.p.\ in\ D.$

$Assume:\ \forall t \in [0,1]:$

$i)\ u \ge v_0\ in\ D$

$ii)\ u \ge v_t\ on\ \partial D,\ u > v_t\ on\ \{v_t > 0\} \cap \partial D$

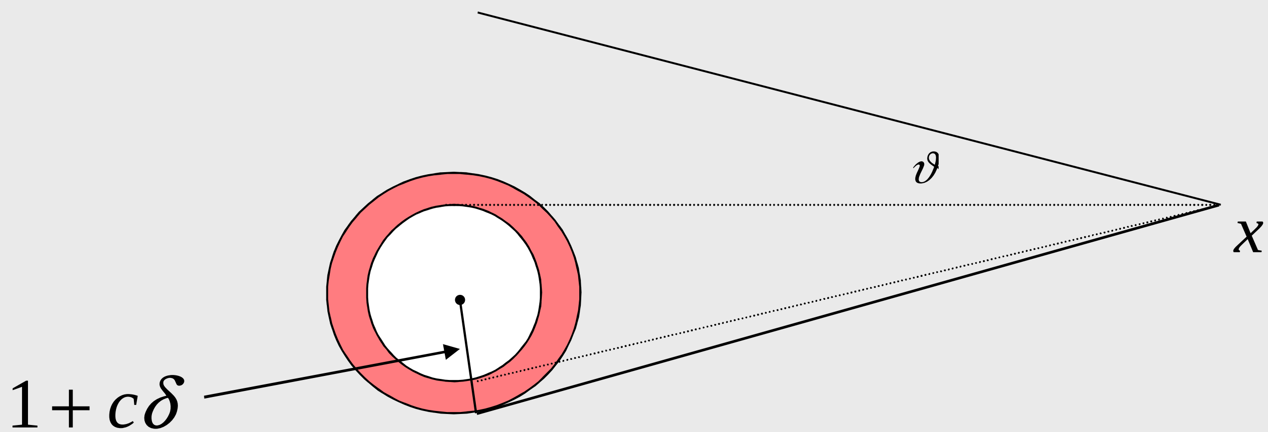
$iii)\ \forall y \in F(v_t)\ is\ right-regular$

$iv)\ \{v_t > 0\}\ moves\ continuously$

$$Then,\ u \ge v_t\ in\ D,\ \forall t \in [0,1].$$

$$v_{\varepsilon\varphi_1}(x) = \sup_{B_{\varepsilon\varphi_1(x)}(x)} u(y-\sigma) \le u(x)$$

$in\ B_{1/2}$ $\quad \varphi_1(x) \ge 1 + c\delta$



(vague) Idea of the proof. *Non divergence case (b=0)*

$$v_g(x_0) = u\left(x_0 + g(x_0)\eta(x_0)\right) \equiv u\left(x^*\right)$$

for some unit vector $\eta(x_0)$

$x_0 = 0, A(0) = I$

$\Longrightarrow$ $$\fint_{\partial B_1} v_g(r\sigma)\,d\sigma \geq v_g(0) + o\left(r^2\right)$$

Select an orthonormal basis $\{e_1, \dots, e_n\}$

Let $e_n = \nabla u\left(x^*\right) / \left|\nabla u\left(x^*\right)\right|$

Choice of the $e_i,\ i = 1,\dots,n-1$

Let $P_n = I - e_n \otimes e_n$ and

$D_T\left(x^*\right) = P_n D^2 u\left(x^*\right) P_n \qquad D_N\left(x^*\right) = D^2 u\left(x^*\right) - D_T\left(x^*\right)$

Let U be the matrix that diagonalizes $D_T\left(x^*\right)$ and let

$\beta_1,\dots,\beta_{n-1},0$ the eigenvalues of $D_T\left(x^*\right)$

$e_1,\dots,e_n$ the columns of U

$$Lu\left(x^*\right) = \sum_{j=1}^{n-1} \alpha^*_{jj}\beta_j\left(x^*\right) + 2\sum_{j=1}^{n-1} \alpha^*_{jn} d^*_{jn}\left(x^*\right) + \alpha^*_{nn} d^*_{nn}$$

$$\xi(x) = e_n + \sum_{i=1}^{n-1} \langle V_i, x \rangle e_i \qquad \eta(x) = \xi(x) / |\xi(x)|$$

$$\eta(x) = e_n + \sum_{i=1}^{n-1} \langle V_i, x \rangle e_i - \frac{1}{2} \sum_{i=1}^{n-1} \langle V_i, x \rangle^2 e_n + o\left(|x|^2\right)$$

$$\Longrightarrow \quad v_g(r\sigma) \geq u\left(r\sigma + g(r\sigma)\eta(r\sigma)\right)$$

$$= u\left(x^*\right) + r \underbrace{l(\sigma)}_{linear} + r^2 \left(\underbrace{M(\sigma)}_{quadratic} + \underbrace{N(\sigma)}_{quadratic} \right) + o\left(r^2\right)$$

$$\fint_{\partial B_1} M(r\sigma) d\sigma = c \left| \nabla u\left(x^*\right) \right| \left\{ \Delta g(0) - c_0 g(0) \sum_{i=1}^{n-1} \left|V_j\right|^2 \right\} + o\left(r^2\right)$$

$$\fint_{\partial B_1} N(r\sigma)\,d\sigma = \sum_{j=1}^{n-1} c_j^2 \beta_j\left(x^*\right) + 2\sum_{j=1}^{n-1} c_{jn} d_{jn}^*\left(x^*\right) + c_n^2 d_{nn}^*$$

$$Lu\left(x^*\right) = \sum_{j=1}^{n-1} \alpha_{jj}^* \beta_j\left(x^*\right) + 2\sum_{j=1}^{n-1} \alpha_{jn}^* d_{jn}^*\left(x^*\right) + \alpha_{nn}^* d_{nn}^*$$

Choose the $V_i,\ i = 1,\ldots,n-1$ in order to reconstruct L and such that

$$\left|V_i\right| \le \frac{c}{g(0)}\left\{\nabla g(0) + \omega\left(\max u / \Lambda\right)\right\}$$

$$\Longrightarrow \qquad \fint_{\partial B_1} N(r\sigma)\,d\sigma = 0$$

Divergence case

To show that:

for each $x \in \Omega^+\left(v_g\right)$, there is $r_0 = r_0\left(x\right)$ such that for every ball $B_r\left(x\right) \subset \Omega^+\left(v_g\right)$, $r \le r_0$ and every $x_0 \in B_r\left(x\right)$

$$\int_{\partial B_r(x)} v_g\left(\sigma\right)\omega_r^{x_0}\left(d\sigma\right) \ge v_g\left(x_0\right)$$

where $\omega_r^{x_0}$ is the L-harmonic measure in $B_r\left(x\right)$

$$v_g\left(x_0\right) = u\left(x_0 + g\left(x_0\right)\eta\left(x_0\right)\right) \equiv u\left(x^*\right)$$

for some unit vector $\eta\left(x_0\right)$

Select an orthonormal basis $\left\{e_1, \ldots, e_n\right\}$ with $e_n = \eta\left(x_0\right)$

Define ($h=\sigma-x_0$):

$$\xi(h)=e_n+\sum_{i=1}^{n-1}\langle V_i,h\rangle e_i \qquad \eta(h)=\xi(h)/|\xi(h)|$$

$$\eta(h)=e_n+\sum_{i=1}^{n-1}\langle V_i,h\rangle e_i-\frac{1}{2}\sum_{i=1}^{n-1}\langle V_i,h\rangle^2 e_n+o(h^2)$$

$$\int_{\partial B_r(x)} v_g(\sigma)\,d\omega_r^{x_0}(\sigma)\geq v_g(x_0)+$$

$$\frac{1}{2}\left|\nabla u(x^*)\right|\int_{\partial B_r(x)}\left[\left\langle 2h,\nabla g(x_0)\right\rangle-g(x_0)\sum_{i}^{n-1}\langle V_i,h\rangle^2+\left\langle D^2 g(x_0)h,h\right\rangle\right]d\omega_r^{x_0}+$$

$$\frac{1}{2}\sum_{i,j}^{n-1}D^2u(x^*)\int_{\partial B_r(x)}a_i a_j\,d\omega_r^{x_0}+\nabla u(x^*)\int_{\partial B_r(x)}h\,d\omega_r^{x_0}+o(r^2)$$

where

$$a_i=\left\langle g(x_0)V_i+e_i,h\right\rangle \quad i=1,\ldots,n.$$