Pathwise uniqueness for singular SDEs driven by stable processes

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Introduction

We prove pathwise uniqueness for the SDE

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + L_t, \qquad x \in \mathbb{R}^d, \;\; t \geqslant 0, \tag{1}$$

assuming

(H1) $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and β-Hölder continuous, (i.e. $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$; $\|b\|_\beta = \|b\|_0 + [b]_\beta < +\infty$),

$$[b]_\beta = \sup_{x \neq y,\; x,y \in \mathbb{R}^d} \frac{|b(x) - b(y)|}{|x-y|^\beta}, \quad \beta \in (0,1).$$

(H2) $L = (L_t)$ is a non-degenerate d-dimensional symmetric α-stable Lévy process, $d \geqslant 1$.

(H3) $\alpha \in [1,2)$ and $\beta > 1 - \frac{\alpha}{2}$.

Pathwise uniqueness : fix a stochastic basis

$$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$$

on which L is defined and consider two d-dimensional stochastic processes $X = (X_t)$ and $Y = (Y_t)$ defined and adapted on the fixed stochastic basis.

If X and Y solve (1), for any $\omega \in \Omega$, $\mathbb{P}$-a.s., then we have, for any $t \geqslant 0$,

$$X_t = Y_t, \quad \mathbb{P} - a.s.$$

Remark 1 The result is due to the non-degeneracy of the noise L in an essential way.

Indeed if we put $L = 0$ then uniqueness fails even if $d = 1$.

Remark 2 Non-degeneracy of the noise

$\Longrightarrow$ existence of a "regular solution" u to a deterministic Kolmogorov equation associated to the SDE

$\Longrightarrow$ uniqueness for the SDE by comparing $u(X_t)$ and $u(Y_t)$.

References (when L is a Wiener process)

Recent interest in understanding pathwise uniqueness for SDEs when b is not Lipschitz continuous or, more generally, when

b is singular enough so that the deterministic equation with $L = 0$ is not well-posed.

- A.K. Zvonkin : Mat. Sb. (N.S.) (1974) [$b \in L^\infty(\mathbb{R})$, i.e., $d = 1$]

- A.J. Veretennikov, Mat. Sb., (N.S.) (1980) [$b \in L^\infty(\mathbb{R}^d)$ (for any $d \geqslant 1$)].
Note that $1 - \alpha/2 \to 0$ as $\alpha \to 2$.

- I. Gyöngy and T. Martinez : Czechoslovak Math. J. (2001) [$b \in L^{2d+2}_{loc}(\mathbb{R}^d)$ and ...]

- N.V. Krylov and M. Röckner : Probab. Theory Relat. Fields (2005) [$b \in L^p_{loc}(\mathbb{R}^d)$ when $p > d$ and ...]

- X. Zhang : Stochastic Process. Appl. (2005) [multiplicative noise].

- F. Flandoli, M. Gubinelli and E. P. : Invent. Math. (2010) [$b \in C^\beta$; uniqueness for stochastic transport equation].

- G. Da Prato and F. Flandoli : J. Funct. Anal. (2010) [pathwise uniqueness for SDEs in Hilbert spaces].

What happens when L is not a Wiener process ?

If L is is not a Wiener process but more generally is a Lévy process, the situation changes even when $\mathbb{R}^d = \mathbb{R}$.

Recall that a d-dimensional stochastic process L defined on a fixed stochastic basis is a Lévy process if it is continuous in probability, it has stationary increments, càdlàg trajectories, $L_t - L_s$ is independent of $\mathcal{F}_s$, $0 \leqslant s \leqslant t$, and $L_0 = 0$.

A Lévy process L can be characterized by its symbol (or exponent) $\psi : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}[e^{i\langle L_t, h\rangle}] = e^{-t\psi(h)}, \quad t \geqslant 0,\ h \in \mathbb{R}^d,$$

(here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^d$; we also use the notation "$\cdot$"). We refer to [Sato 1999] and [Applebaum 2004].

When $\mathbb{R}^d = \mathbb{R}$, L is a symmetric α-stable Lévy process if

$$\psi(h) = c_\alpha |h|^\alpha, \quad h \in \mathbb{R},$$

for some $\alpha \in (0, 2)$.

H. Tanaka, M. Tsuchiya and S. Watanabe proved in J. Math. Kyoto Univ. (1974) when $d = 1$:

$$\alpha < 1 \text{ and } b \in C_b^{\beta}(\mathbb{R}) \text{ with } \alpha + \beta < 1 \implies \text{pathwise uniqueness fails.}$$

They show this even with the initial condition $x = 0$.

Moreover, when $\alpha \geqslant 1$ and $b \in C_b(\mathbb{R})$ they prove pathwise uniqueness.

Two examples of Lévy processes which verifies (H2) (to have in mind)

Example 1. $L = (L_t)$ is a d-dimensional Lévy process, $d \geqslant 1$, such that

$$\mathbb{E}[e^{i\langle L_t, h\rangle}] = e^{-c_\alpha t|h|^\alpha}, \quad h \in \mathbb{R}^d, \ \ t \geqslant 0, \ \ \alpha \in (0,2);$$

L is called the standard (symmetric and rotationally invariant) α-stable process.

The generator of L is $\mathcal{L} = -(-\triangle)^{\alpha/2}$; for any $x \in \mathbb{R}^d, f \in C_0^\infty(\mathbb{R}^d)$,

$$-(-\triangle)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \big(f(x+y) - f(x) - 1_{\{|y| \leqslant 1\}} \langle y, Df(x) \rangle \big) \frac{\tilde{c}_\alpha}{|y|^{d+\alpha}} dy. \tag{2}$$

The measure ν with density $\frac{\tilde{c}_\alpha}{|y|^{d+\alpha}}$ is the Lévy measure of the process L.

Example 2. L is a Lévy process such that

$$\mathbb{E}[e^{i\langle L_t,h\rangle}] = e^{-c_\alpha t(|h_1|^\alpha+\ldots+|h_d|^\alpha)}, \quad h \in \mathbb{R}^d, \quad t \geqslant 0, \quad \alpha \in (0,2).$$

In this case $L = (L_t^1, \ldots, L_t^d)$, where $L^1, \ldots, L^d$ are independent one-dimensional symmetric α-stable processes.

The generator of L is

$$\mathcal{L} = -\sum_{k=1}^{d} (-\partial^2_{x_k x_k})^{\alpha/2},$$

i.e., for any $x \in \mathbb{R}^d, f \in C_0^\infty(\mathbb{R}^d)$,

$$\mathcal{L}f(x) = \sum_{k=1}^{d} \int_{\mathbb{R}} [f(x+se_k) - f(x) - 1_{\{|s|\leqslant 1\}}\, s\, \partial_{x_k} f(x)] \, \frac{c_\alpha}{|s|^{1+\alpha}} ds.$$

Martingale problems for SDEs driven by $(L_t^1, \ldots, L_t^d)$ have been recently studied in [Bass-Chen 2006].

Explanation of hypothesis (H2)

A (positive) Borel measure γ on $\mathbb{R}^d$ is symmetric if $\gamma(D) = \gamma(-D)$, $D \in \mathcal{B}(\mathbb{R}^d)$.

Let $\alpha \in (0,2)$. In the SDE we consider a d-dimensional Lévy process $L = (L_t)$ which is also *symmetric α-stable.* This means that

$$\mathbb{E}[e^{i\langle L_t,h\rangle}] = e^{-t\psi(h)}, \quad \psi(h) = -\int_{\mathbb{R}^d} \Big(e^{i\langle h,y\rangle} - 1 - i\langle h,y\rangle \, 1_{\{|h|\leqslant 1\}}(y) \Big) \nu(dy), \qquad (3)$$

$h \in \mathbb{R}^d$, $t \geqslant 0$, where for $D \subset \mathbb{R}^d$, 1_D is the indicator function of D ($1_D(x) = 1 \iff x \in D$) and ν is a Borel measure (the Lévy measure of L) such that

$$\nu(D) = \int_{\S} \mu(d\xi) \int_0^\infty 1_D(r\xi) \frac{dr}{r^{1+\alpha}}, \quad D \in \mathcal{B}(\mathbb{R}^d), \qquad (4)$$

for some symmetric, non-zero finite Borel measure μ concentrated on $S = \{y \in \mathbb{R}^d \ : \ |y| = 1\}$.

Formula (3) is the Lévy-Khintchine formula for L. By (4), we rewrite (3) as

$$\psi(h) = -\int_{\mathbb{R}^d} \big(\cos(\langle h,y\rangle) - 1 \big) \nu(dy) = c_\alpha \int_S |\langle h,\xi\rangle|^\alpha \mu(d\xi), \ \ h \in \mathbb{R}^d.$$

The measure μ is called the spectral measure of L.

We assume that *the stable process L is not degenerate*; this means that (see [Sztonik 2010])

The support of the spectral measure μ is not contained in a proper linear subspace of $\mathbb{R}^d$.

It is not difficult to show that this is equivalent to

$$\psi(u) \geqslant C_\alpha |u|^\alpha, \quad u \in \mathbb{R}^d,$$

for some positive constant C_α.

Remark that in Example 2, $\mu = \sum_{k=1}^d (\delta_{e_k} + \delta_{-e_k})$, where (e_k) is the canonical basis in $\mathbb{R}^d$.

Finally the infinitesimal generator $\mathcal{L}$ of the process L is given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} \big(f(x+y) - f(x) - 1_{\{|y| \leqslant 1\}} \langle y, Df(x) \rangle\big)\, \nu(dy), \quad f \in C_0^\infty(\mathbb{R}^d).$$

Strategy of proof

Remark that existence of solution follows by a compactness argument (Ascoli-Arzela's theorem)

The proof of uniqueness consists of two main parts:

I Part. An Itô-Tanaka trick (as in [Fandoli-Gubinelli-P. 2010])

II Part. Analytic regularity results (Schauder estimates) for a nonlocal Kolmogorov operator associated to L.

The Itô-Tanaka trick

Poisson Random measures. The Poisson random measure N associated to the α-stable process L is

$$N((0,t] \times U) = \sum_{0<s\leqslant t} 1_U(\triangle L_s) = \sharp\{0 < s \leqslant t \ : \ \triangle L_s \in U\},$$

for any $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $t > 0$.

Here $\triangle L_s = L_s - L_{s-}$ is the jump size of L at time $s > 0$. The compensated Poisson random measure $\tilde{N}$ is defined by

$$\tilde{N}((0,t] \times U) = N((0,t] \times U) - t\nu(U),$$

where ν is the Lévy measure given in (4).

Let $X = X_t$ be a solution (starting at $x \in \mathbb{R}^d$). For a deterministic function $f : \mathbb{R}^d \to \mathbb{R}$ we can define

$$\int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [f(X_{s-} + x) - f(X_{s-})]\tilde{N}(ds, dx)$$

if $\int_0^t \int_{\mathbb{R}^d\setminus\{0\}} |f(X_{s-} + x) - f(X_{s-})|^2 \nu(dx) < \infty$.

I Step. Let $X = X_t$ be a solution. Applying Itô formula we get

$$\begin{aligned} & f(X_t) - f(x) \\ & = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f(X_{s-} + x) - f(X_{s-})]\, \tilde{N}(ds, dx) \\ & + \int_0^t ds \int_{\mathbb{R}^d} [f(X_{s-} + x) - f(X_{s-}) - 1_{\{|x| \leqslant 1\}}\, x \cdot Df(X_{s-})] \nu(dx) \\ & \qquad + \int_0^t b(X_s) \cdot Df(X_s) ds. \end{aligned} \tag{5}$$

Usually this formula require f of class C^2 (see [Applebaum 2004]).

However for α-stable processes L

$$\text{Itô formula holds when} \quad f \in C_b^{1+\gamma}(\mathbb{R}^d), \quad 1 + \gamma > \alpha$$

II Step. Let us consider the following resolvent equation on $\mathbb{R}^d$

$$\lambda u - \mathcal{L}u - Du \cdot b = b, \tag{6}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is given in the SDE, $\mathcal{L}$ is the generator of L and $\lambda > 0$.

The equation must be understood componentwise, i.e.,
$\lambda u_i - \mathcal{L}u_i - b \cdot Du_i = b_i, i = 1, \ldots, d.$

Assume that for some $\lambda > 0$ there exists a solution $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to (6) (with $\gamma \in [0,1]$) such that $\boxed{1 + \gamma > \alpha}$ then by Itô formula

$$u(X_t) - u(x) = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})] \tilde{N}(ds, dx)$$
$$+ \int_0^t \big(\mathcal{L}u(X_s) + Du(X_s)b(X_s)\big) ds$$

and using that u solves (6), i.e., $\boxed{\mathcal{L}u + b \cdot Du = \lambda u - b}$, we can replace

$$\int_0^t \mathcal{L}u(X_s)ds + \int_0^t Du(X_s)b(X_s)ds$$

with $-\int_0^t b(X_s)ds + \lambda \int_0^t u(X_s)ds = x - X_t + L_t + \lambda \int_0^t u(X_s)ds$ (see the SDE) and obtain the identity, $\mathbb{P}$-a.s., $t \geqslant 0$,

$$u(X_t) - u(x)$$
$$= x - X_t + L_t + \lambda \int_0^t u(X_s)ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})] \tilde{N}(ds, dx).$$

III Step. Let now X and Y be two solutions (starting at $x \in \mathbb{R}^d$). By the previous identity , $\mathbb{P}$-a.s.,

$$X_t - Y_t = [u(Y_t) - u(X_t)]$$
$$+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx)$$
$$+ \lambda \int_0^t [u(X_s) - u(Y_s)] ds.$$

Now in order to apply the Gronwall lemma to $\mathbb{E}|X_t - Y_t|^2$ and get uniqueness we need

$$\boxed{\|Du_\lambda\|_0 < 1 \quad \text{and} \quad 2\gamma > \alpha.}$$

The condition $2\gamma > \alpha$ (which implies $1+\gamma > \alpha$) is needed since

$$\mathbb{E}\Big|\int_0^t\int_{\{|x|\leqslant 1\}}[u(X_{s-}+x)-u(X_{s-})-u(Y_{s-}+x)+u(Y_{s-})]\tilde{N}(ds,dx)\Big|^2$$

$$=\mathbb{E}\int_0^t\int_{\{|x|\leqslant 1\}}|u(X_{s-}+x)-u(X_{s-})-u(Y_{s-}+x)+u(Y_{s-})|^2\nu(dx)$$

$$\leqslant c\|u\|_{1+\gamma}^2\int_0^t E|X_s-Y_s|^2 ds\int_{\{|x|\leqslant 1\}}|x|^{2\gamma}\nu(dx)$$

and $\int_{\{|x|\leqslant 1\}}|x|^{2\gamma}\nu(dx)<\infty$ only if $2\gamma>\alpha$. I have also used

Lemma

Let $\gamma\in[0,1]$ and $f\in C_b^{1+\gamma}(\mathbb{R}^d)$. Then for any $u,v\in\mathbb{R}^d$, $x\in\mathbb{R}^d$, with $|x|\leqslant 1$, we have

$$|f(u+x)-f(u)-f(v+x)+f(v)|\leqslant c_\gamma\|f\|_{1+\gamma}|u-v|\,|x|^\gamma,\quad \textit{with } c_\gamma=3^{1-\gamma}.$$

Analytic results

We deal with

$$\lambda u - \mathcal{L}u - b \cdot Du = g$$

where $\lambda > 0$, $g \in C_b^\beta(\mathbb{R}^d)$ and $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$.

We can study this problem when $\alpha \geqslant 1$. The more difficult case is $\alpha = 1$.

We will use the following properties of L (μ_t denotes the law of L_t).

(a) $\mu_t(A) = \mu_1(t^{-1/\alpha}A)$, for any $A \in \mathcal{B}(\mathbb{R}^d)$, $t > 0$;

(b) μ_t has a density p_t; moreover $p_t \in C^1(\mathbb{R}^d)$ and its spatial derivative $Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ (follows from the non-degeneracy assumption)

(c) for any $\sigma > \alpha$, we have $\displaystyle\int_{\{|x| \leqslant 1\}} |x|^\sigma \nu(dx) < \infty$.

We first prove Schauder estimates for the resolvent equation with b constant, i.e.,

$$b(x) = b_0, \quad x \in \mathbb{R}^d.$$

We stress that the Schauder constant c in (7) is independent of b_0; this fact is needed to treat $\alpha = 1$ when b is variable.

Theorem

Assume (H1), (H2). Let $\boxed{\alpha \in (0,2)}$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha + \beta < 2$. *Then, for any* $\lambda > 0$, $b_0 \in \mathbb{R}^d$, $g \in C_b^\beta(\mathbb{R}^d)$, *there exists a unique solution* $u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ *to the equation*

$$\lambda u - \mathcal{L}u - b_0 \cdot Du = g$$

on $\mathbb{R}^d$. *In addition there exists a constant* c *independent of* g, u, b_0 *and* $\lambda > 0$ *such that*

$$\lambda \|u\|_0 + \lambda^{\frac{\alpha+\beta-1}{\alpha}} \|Du\|_0 + [Du]_{\alpha+\beta-1} \leqslant c\|g\|_\beta. \tag{7}$$

In the proof we first introduce the α-stable Markov semigroup (P_t) acting on $C_b(\mathbb{R}^d)$ and associated to $\mathcal{L} + b_0 \cdot Du$, i.e.,

$$P_t f(x) = \int_{\mathbb{R}^d} f(z + tb_0)\, p_t(z - x) dz, \quad t > 0, \; f \in C_b(\mathbb{R}^d), \; x \in \mathbb{R}^d,$$

where p_t is the density of L_t, and $P_0 = I$.

Then we consider the bounded function $u = u_\lambda$,

$$u(x) = \int_0^\infty e^{-\lambda t} P_t g(x) dt, \quad x \in \mathbb{R}^d. \tag{8}$$

and show that u belongs to $C_b^{\alpha+\beta}(\mathbb{R}^d)$, verifies (7) and solves the equation. To this purpose we also need a maximum principle.

We consider now $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$.

Theorem

Assume (H1) and (H2). Let $\alpha \geqslant 1$ *and* $\beta \in (0,1)$ *be such that* $1 < \alpha + \beta < 2$. *Then, for any* $\lambda > 0$, $g \in C_b^\beta(\mathbb{R}^d)$, *there exists a unique solution* $u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ *to*

$$\lambda u - \mathcal{L} u - b \cdot Du = g$$

on $\mathbb{R}^d$. *Moreover, for any* $\omega > 0$, *there exists* $c = c(\omega)$, *independent of* g *and* u, *such that*

$$\lambda \|u\|_0 + [Du]_{\alpha+\beta-1} \leqslant c\|g\|_\beta, \quad \lambda \geqslant \omega. \tag{9}$$

Finally, we have $\lim_{\lambda\to\infty} \|Du_\lambda\|_0 = 0$.

Remark Note that $\gamma = \alpha + \beta - 1$. Hence the condition $2\gamma > \alpha$ in the Itô-Tanaka trick becomes

$$\beta > 1 - \alpha/2.$$

Note that the last assertion follows from (9). Indeed we obtain, for $\lambda \geqslant \omega$,

$$\|Du_\lambda\|_0 \leqslant N[Du_\lambda]_{\alpha+\beta-1}^{\frac{1}{\alpha+\beta}} \, \|u_\lambda\|_0^{1-\frac{1}{\alpha+\beta}} \leqslant N\tilde{c}\, \lambda^{-\frac{\alpha+\beta-1}{\alpha+\beta}} \, \|g\|_\beta,$$

where $\tilde{c} = \tilde{c}(\omega)$ and $N = N(d, \alpha, \beta)$.

The proof is based on a-priori estimates which allow to use the well known continuity method.

Such estimates are not difficult when $\alpha > 1$. Note that in the case $\mathcal{L} = (-\triangle)^{\alpha/2}$ if $\alpha > 1$ the term

$$b \cdot Du \quad \text{is "of lower order" with respect to } -(-\triangle)^{\alpha/2}.$$

The critical case is $\alpha = 1$ where we need to use a localization procedure.

This is based on the fact that in the first theorem (where $b = b_0$) the Schauder constant is independent on b_0.

When b is variable we can not show existence of $C_b^{\alpha+\beta}$-solutions when $\alpha < 1$.

To see this, write

$$\lambda u(x) - \mathcal{L}u(x) = g(x) + b(x) \cdot Du(x).$$

We get the a-priori estimate

$$[Du]_{\alpha+\beta-1} \leqslant C\|g\|_\beta + C\|b\|_\beta \|Du\|_0 + C\|b\|_0 [Du]_\beta$$

but we cannot continue, since $\alpha < 1$ gives $Du \in C_b^\theta$ with $\theta = \alpha + \beta - 1 < \beta$.

Roughly speaking, when $\alpha < 1$, the perturbation term $b \cdot Du$ seems to be stronger than $\mathcal{L}$.

Thus pathwise uniqueness when $\alpha < 1$ is an open problem ($d > 1$).

Conclusion.

My complete result on the SDE is the following.

Theorem (P. 2010; Preprint Arxiv)

Let L be a symmetric α-stable process with $\alpha \in [1,2)$, satisfying (H1)-(H3). Assume that $b \in C_b^{\beta}\left(\mathbb{R}^d;\mathbb{R}^d\right)$ for some $\beta \in (0,1)$ such that

$$\beta > 1 - \frac{\alpha}{2}.$$

Then pathwise uniqueness holds for equation (1). Moreover, if $X^x = (X_t^x)$ denotes the solution starting at $x \in \mathbb{R}^d$, we have:

(i) for any $t \geqslant 0$, $p \geqslant 1$, there exists a constant $C(t,p) > 0$ (depending also on α, β and $L = (L_t)$) such that

$$\mathbb{E}[\sup_{0\leqslant s\leqslant t} |X_s^x - X_s^y|^p] \leqslant C(t,p)\, |x-y|^p, \quad x, y \in \mathbb{R}^d;$$

(ii) for any $t \geqslant 0$, the mapping: $x \mapsto X_t^x$ is a homeomorphism from $\mathbb{R}^d$ onto $\mathbb{R}^d$, $\mathbb{P}$-a.s.;

(iii) for any $t \geqslant 0$, the mapping: $x \mapsto X_t^x$ is a C^1-function on $\mathbb{R}^d$, $\mathbb{P}$-a.s..