REGULARITY FOR SINGULAR RISK-NEUTRAL VALUATION EQUATIONS

Kolmogorov Equations in Physics and Finance

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Pricing Financial Derivatives

- In valuing financial derivatives it is obviously preferable to have closed-form - or as near as possible to closed-form - expressions for the price of the security.
- This is the reason for the success of **affine models**. The price can be found by solving a system of ODEs and then inverting Fourier transforms

  (see, [**Duffie, Filipovich, and Schachermayer (2003)**]).
- Many pricing problems, including some classical ones, cannot be formulated by affine models:

  The Arithmetic Asian option in the Heston model

  The Hobson-Rogers model

  The Bates model with downward jumps in the volatility

The Problem

- When the problem at hand does not fit in the class of affine models, the no-arbitrage price can be computed as the solution of

$$(\mathbf{P}) \begin{cases} \partial_t u(t,x) + Lu(t,x) - c(x)u(t,x) = f(t,x), & (t,x) \in (0,T) \times D, \\ u(T,x) = \phi(x), & x \in D, \end{cases}$$

$$Lg(x) = \nabla g(x) b(x) + \frac{1}{2} \mathrm{tr}\left(\nabla^2 g(x)\sigma(x)\sigma^T(x)\right) + \int_D \left[g(z) - g(x)\right] m(x, dz).$$

- $D \subset \mathbb{R}^d$, b is the drift of the state process X_t, $a = \sigma\sigma^\top$ is the diffusion matrix, c is the "discount rate function";
- ϕ is the payoff and f is a continuous yield or a running cost;
- m includes the jump intensity and probability distribution of X_t after the jump.

The valuation equation may exhibit, often simultaneously, several indeterminacies and degeneracies such as:

- The diffusion matrix a is singular on the boundary of D, or is even identically zero in some direction: The former singularity arises in some stochastic volatility models, like the Heston model; the latter is the case of Asian options, of some path dependent volatility models, like the Hobson-Rogers model, or of models where some components are pure jump.
- The drift b and the matrix σ are not Lipschitz continuous up to the boundary of D: This occurs, e.g., whenever some components are **square root diffusions** (CIR or Heston models).
- The coefficients b and a are fast growing near the boundary or at infinity: The latter occurs, e.g., in Asian option pricing with the Heston model and in the Hobson-Rogers model.

. . . Continue

- The jump intensity is not bounded: This occurs, e.g., in some generalizations of affine models like the Bates model with downward jumps of the volatility.
- The state space D has a boundary but no boundary conditions are specified: This is the case in most models where D has a boundary.

Examples

▶ **Asian option pricing:** The payoff of an Asian option is a function of

$$\frac{1}{T}\int_0^T S_t\,dt \quad (\textbf{Arithmetic}) \qquad \frac{1}{T}\int_0^T \log S_t\; dt, \quad (\textbf{Geometric})$$

In addition the payoff may depend on S_T as well.

The standard approach is to introduce the process A_t satisfying,

$$dA_t = S_t\,dt, \qquad \text{or} \qquad dA_t = \log S_t\,dt$$

• A_t is purely deterministic, given S_t, the last row and the last column of the diffusion matrix $\sigma\sigma^\top$ are always identically zero

• The theory of classical solutions of PDE's does not apply, nor does, in general, the standard theory of viscosity solutions.

▶ The Heston model:

In the Heston model the asset price, S_t, and the stochastic volatility, V_t, follow

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{V_t}S_t\,dW_t^1, \\ dV_t = \beta(\overline{v} - V_t)\,dt + \sigma_0\sqrt{V_t}\,\left[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right]. \end{cases}$$

where

- W^1 and W^2 are independent Brownian motions;
- r is a constant interest rate and $\rho \in [-1, 1]$.
- β, $\overline{v}$ and σ_0 are positive parameters, satisfying the **Feller** condition $(2\beta\overline{v} > \sigma_0^2)$.

$$a(S,V) = \begin{pmatrix} VS^2 & \sigma_0\rho VS \\ \sigma_0\rho VS & \sigma_0^2 V \end{pmatrix}, \qquad b(S,V) = \begin{pmatrix} rS \\ \beta(\overline{v} - V) \end{pmatrix}.$$

Viscosity Solutions

- In the presence of these features, existence, uniqueness or regularity of solutions may
- It is common practice in mathematical finance to develop numerical
- The classical theory of Sobolev spaces applies only if the diffusion matrix a is uniformly nonsingular in the state space D.
- In contrast the theory of viscosity solutions allows to deal with singular diffusion matrices.
- Viscosity solutions are continuous functions (a priori not differentiable) and are well suited to be computed

- Originally developed for linear and nonlinear Partial Differential Equations (Crandall, Ishii and Lions (1992), Bardi and Capuzzo Dolcetta (1997), and Fleming and Soner (1993)), the theory of viscosity solutions has later been extended to (linear and nonlinear) Partial Integro-Differential Equations (e.g. Alvarez and Tourin (1996), Pham (1998), Jakobsen and Karlsen (2005-2006) , Alibaud (2007), and references therein).
- The existing results are not sufficient to deal with the above described features, even in the linear case we are considering here, and even in the purely differential case.
- Previous works assume globally Lipschitz coefficients with sublinear growth or they assume bounded, uniformly continuous coefficients.
- Alvarez and Tourin consider an operator of our form with a uniformly bounded measure m.

- Other works consider more general operators - integral operators corresponding to Levy jump processes, or even more general non local operators.
- However, when these operators are specialized to our form, the measure m is assumed to be uniformly bounded.
- Di Francesco, Pascucci, Polidoro (2006), Pascucci (2008), Frentz, Nyström and Pascucci (2009) investigate the purely differential degenerate valuation equations with a special structure that ensures that the equation is hypoelliptic.
- There is a well developed literature on PDE methods with Sobolev and Hölder spaces suitably defined, which covers at least the case of Heston, Asian and path dependent models in the purely diffusive case, for instance: Bramanti, Cerutti, Manfredini (1996), Lunardi (1998), Di Francesco, Polidoro (2006).

Remark on the structure of a

Remark: If there exists $\lambda > 0$ and a $n \times n$ matrix-valued function $A(x)$, $n \leq d$ such that

$$a(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda^{-1}|\eta|^2 < \langle A(x)\eta, \eta\rangle < \lambda|\eta|^2, \quad \forall\, \eta \in \mathbb{R}^n$$

that is a is uniformly positive definite only on a linear subspace of $\mathbb{R}^d$ (or on the intersection of such a linear subspace with an orthant) and identically null on its orthogonal complement, then this special condition is not sufficient to cover the case of Geometric and Arithmetic Asian options under the Heston model.

- In fact the specific cases of the Arithmetic Asian floating-strike put and fixed-strike call in the Black-Scholes model are solved in Barucci, Polidoro, Vespri (2001) with a different approach.

- In the framework of classical solutions, Janson and Tysk (2006) prove existence and uniqueness of the solution in $D = (0; +\infty)^d$, assuming that σ is locally Lipschitz continuous in D, has sublinear growth and full rank in D, but its i-th row vanishes on the set $x_i = 0$, for all i's, and the Lipschitz condition does not necessarily hold on the boundary.

- Ekström and Tysk (2010) study classical solutions of B&S equation in SV models when the boundary may be attainable. Precisely, the operator has the form

$$L = \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + \rho\sigma\sqrt{y} x \frac{\partial^2}{\partial x \partial y} + \frac{\sigma^2(y)}{2} \frac{\partial^2}{\partial y^2} + \beta(y) \frac{\partial}{\partial y}$$

- $\beta \in \mathcal{C}^{1+\alpha}([0,+\infty))$, $\beta(0) \geq 0$.
- $\sigma^2 \in \mathcal{C}^{1+\alpha}([0,+\infty))$, $\sigma(0) = 0$, $\sigma(y) > 0$ for any $y > 0$.
- $|\beta(y)| + \sigma(y) \leq C(1+y)$ holds for all $y \geq 0$.
- The payoff function $\phi \in \mathcal{C}_b([0,+\infty)) \cap \mathcal{C}^2([0,+\infty))$.
- $x\phi'(x)$ and $x^2\phi''(x)$ are bounded.

Then, the pricing problem

$$\begin{cases} \partial_t u + Lu = 0, \quad \text{in } (0,+\infty)^2 \times [0,T), \\ u(t,0,y) = \phi(0), \\ \partial_t u(t,x,0) + \beta(0)\partial_y u(t,x,0) = 0, \\ u(T,x,y) = \phi(x), \end{cases}$$

has a unique classical solution:

$$u \in \mathcal{C}^{2,2,1}((0,+\infty)^2 \times [0,T)) \cap \mathcal{C}^{0,1,1}((0,+\infty) \times [0,+\infty) \times [0,T))$$

Assumptions

- D is a (possibly unbounded) starshaped open subset of $\mathbf{R}^d$.
- $\mathcal{M}(D)$ is the space of finite Borel measures on D, endowed with the weak convergence topology.

Assumption 1. $\sigma : D \to \mathbf{R}^{d\times d}$ and $b : D \to \mathbf{R}^d$ are locally Lipschitz continuous on D.

For the full integro-differential case, we assume in addition:

Assumption 2. $a_{i,j} \in \mathcal{C}^2(D)$, $m : D \to \mathcal{M}(D)$ is continuous and

$$\sup_{x\in D} \left| \int_D g(z) m(x, dz) \right| < \infty, \qquad \forall\, g \in \mathcal{C}_c(D).$$

Remark: For $d = 1$, the **Assumption 1** can be weakened:
$\sigma : D \to \mathbf{R}$ Hölder continuous of order $1/2$ on compact subsets of D.

A Structural Condition

Assumption 3 (Lyapunov type condition). There exists a nonnegative function $V \in \mathcal{C}^2(D)$, such that

$$\int_D V(z)m(x,dz) < +\infty, \quad LV(x) \leq C\left(1+V(x)\right), \ \ \forall x \in D,$$

$$\lim_{x\in D,\, x\to\overline{x}} V(x) = +\infty, \forall \ \overline{x} \in \partial D, \qquad \lim_{x\in D,\, |x|\to+\infty} V(x) = +\infty.$$

This ensures that the stochastic process does not blow up in finite time and does not reach the boundary of D.

Remarks:

- No growth condition on the coefficients is needed.
- The function V also determines the growth rate that we can allow for the data.

Assumption 4. $f \in \mathcal{C}((0,T) \times D)$, c, $\phi \in \mathcal{C}(D)$. c is bounded from below. There exists a strictly increasing function
$\varphi : [0, +\infty) \to [0, +\infty)$, such that

$$s \mapsto s\varphi(s) \quad \text{is convex,} \qquad \lim_{s \to +\infty} \varphi(s) = +\infty,$$

$$(s_1 + s_2)\varphi(s_1 + s_2) \leq C\left(s_1\varphi(s_1) + s_2\varphi(s_2)\right), \quad \forall\, s_1, s_2 \geq 0,$$

such that the following hold on $(0,T) \times D$:

$$|f(t,x)|\varphi(|f(t,x)|) + |\phi(x)|\varphi(|\phi(x)|) \leq C\left(1 + V(x)\right).$$

Remarks:

- Possible choices: $\varphi(s) = s^\alpha$ or $\varphi(s) = \log(s + 1 + \alpha)$, with $\alpha > 0$.
- Under sublinear/subquadratic growth conditions on the coefficients, ϕ and f are allowed to have polynomial growth of order $q < 2$: $|f(t,x)| + |\phi(x)| \leq C(1 + |x|^q)$.

▶ **Arithmetic Asian *floating-strike* put in the Heston model**:

$$\begin{cases} dS_t = rS_t\,dt + \sqrt{V_t}S_t\,dW_t^1, \\ dV_t = \beta(\overline{v} - V_t)\,dt + \sigma_0\sqrt{V_t}\,\left[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right], \end{cases}$$

$$\phi(S_T, A_T) = \left(S_T - \frac{1}{T}A_T\right)_+, \qquad dA_t = S_t dt.$$

- We take $D = (0,\infty)^3$ and assume that $2\beta\,\overline{v} > \sigma_0^2$.

$$\begin{cases} \partial_t u(t,s,v,a) + Lu(t,s,v,a) - r\,u(t,s,v,a) = 0, & \text{in } (0,T)\times D, \\ u(T,s,v,a) = \phi(s,a), & \text{in } D, \end{cases}$$

$$Lg = rs\partial_s g + \beta(\overline{v} - v)\,\partial_v g + s\,\partial_a g + \frac{vs^2}{2}\partial_s^2 g + \frac{\sigma_0^2 v}{2}\partial_v^2 g.$$

The following function satisfies **Assumption 3**:

$$V(s,v,a) = -\log s - \log v - \log a + s\log(s+3) + s(v+1) + v + a.$$

▶ **Geometric Asian Option in the Heston model**: $Z_t = \log(S_t)$, v_t is the volatility process, $G_t = \int_0^t Z_s ds$, $\phi(Z, v, G) = (K - e^{G/T})^+$,

$$dZ_t = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_t^1, \quad dv_t = k(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^2, \quad v_0 > 0.$$

$$\partial_t u + \frac{v}{2}\partial_Z^2 u + \frac{\sigma^2 v}{2}\partial_v^2 u + \left(r - \frac{v}{2}\right)\partial_Z u + k(\theta - v)\partial_v u + Z\partial_G u - ru = 0.$$

$$D = \mathbf{R}\times]0, +\infty[\times\mathbf{R}, \quad \sigma = \begin{bmatrix} \sqrt{v} & 0 \\ 0 & \sigma\sqrt{v} \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} r - \frac{v}{2} \\ k(\theta - v) \\ Z \end{bmatrix}.$$

$$\Longrightarrow V(Z, v, G) = v^{-a} + v^2 + Z^2 + G^2,\ a = \frac{2k\theta}{\sigma^2} - 1 > 0 \ (\Longleftarrow \ v_t > 0).$$

$$LV(Z, v, G) \leq C(1 + Z^2 + v^2 + G^2) + av^{-a-1}\left[\frac{\sigma^2}{2}(a+1) - k\theta\right] + k\theta v^{-a}$$

$$\leq C'(1 + V(Z, v, G)).$$

▶ **Jump-diffusion and stochastic volatility:** Both stochastic volatility and jumps were introduced with the goal of explaining the volatility smile.

• Affine models allow only for nonnegative jumps of the nonnegative components. At least in European option pricing, this is not a problem for the asset price, as it can be viewed as the exponential of the log-return.

• On the contrary, it seems restrictive to impose the volatility to have only upward jumps.

• The following model is proposed in Duffie, Pan and Singleton (2000), except that the jump distribution allows for downward jumps in the volatility:

$$\begin{bmatrix} dY_t \\ dV_t \end{bmatrix} = \begin{bmatrix} \bar{b} - \frac{1}{2}V_t \\ \beta(\bar{v} - V_t) \end{bmatrix} dt + \sqrt{V_t} \begin{bmatrix} 1 & 0 \\ \rho\sigma_0 & \sqrt{1-\rho^2}\sigma_0 \end{bmatrix} dW_t + dZ_t$$

. . . Z is a 2-dimensional, pure jump process with constant mean arrival rate $\overline{\lambda} > 0$ and bivariate jump distribution μ

- Jumps in the log-return alone $\sim \mathcal{N}(m_1, \sigma_1^2)$

- Jumps in the volatility alone $\sim \Gamma(2, \frac{2}{v+m_2})$ (shifted to the left by the value of the volatility at the moment of the jump)

- Common jumps in the log-return and the volatility $\sim \Gamma(2, \frac{2}{v+m_2})$ (shifted to the left by the current volatility value v), and the log-return jump follows $\sim \mathcal{N}(m_{1,c} + \rho_c v', \sigma_{1,c}^2)$, conditionally on a value v' of the volatility jump

The state space is $D = \mathbf{R} \times (0, \infty)$, and we assume $2\,\beta\overline{v} > \sigma_0^2$.

. . . The valuation equation is

$$\begin{cases} \partial_t u(t,y,v) + Lu(t,y,v) - r\,u(t,y,v) = 0, & \text{in } (0,T)\times D, \\ u(T,y,v) = (k-e^y)_+\,, & \text{in } D, \end{cases}$$

$$Lg = \mathcal{A}g + Jg$$

$$\mathcal{A}g(y,v) = \left(\bar{b} - \frac{v}{2}\right)\partial_y\, g(y,v) + \beta(\bar{v}-v)\,\partial_v\, g(y,v) + \frac{v}{2}\,\partial_y^2\, g(y,v)$$
$$+\frac{\sigma_0^2\, v}{2}\,\partial_v^2\, g(y,v) + \frac{1}{2}\,\rho\,\sigma_0\, v\,\partial_{yv}^2\, g(y,v),$$

$$Jg(y,v) = \bar{\lambda}\int_{\mathbf{R}\times(0,+\infty)} g(y+y',v+v')\mu\left((y,v),(dy',dv')\right) - \bar{\lambda}\, g(y,v).$$

- The function

$$V(y,v) = y^2 + v^2 - \log\, v,$$

satisfies **Assumption 3**.

- **Affine Models.** For affine models with affine payoff more explicit methods than solving the valuation equation are available.
- It is worth mentioning that all our assumptions are satisfied by a **conservative regular affine process** whose jump measure has finite second moment:

$$D = (0, +\infty)^{d_0} \times \mathbf{R}^{d-d_0} \qquad b(x) = b^0 + Bx, \qquad a(x) = \alpha^0 + \sum_{h=1}^{d_0} x_h \alpha^h$$

$$m(x, E) = \mu^0(\{z : z + x \in E\}) + \sum_{h=1}^{d_0} x_h \mu^h(\{z : z + x \in E\}),$$
$$\forall \ x \in D, \quad E \in \mathcal{B}(D)$$

The **Assumption 3** is satisfied by

$$V(x) = |x|^2 + \sum_{i=1}^{d_0} (-\log x_i)$$

▶ **SV under the structure of Tysk and Ekström (2010):**

$$L = \frac{1}{2} y x^2 \frac{\partial^2}{\partial x^2} + \rho\sigma\sqrt{y} x \frac{\partial^2}{\partial x \partial y} + \frac{\sigma^2(y)}{2} \frac{\partial^2}{\partial y^2} + \beta(y) \frac{\partial}{\partial y}$$

Let $\overline{y} > 0$ and define $\pi \in \mathcal{C}^2((0, +\infty))$ such that

$$\pi(y) = \int_y^{\overline{y}} \exp\left(\int_r^{\overline{y}} \frac{2\beta(\eta)}{\sigma^2(\eta)} d\eta \right) dr, \qquad \forall\, y < \overline{y}.$$

This is a solution of $L\pi = 0$ on $(0, \overline{y})$.

• If $\pi(y) \to +\infty$ as $y \to 0^+$, then

$$V(x, y) = x \log(x + 3) - \log(x) + x(y + 1) + y + \pi(y),$$

is a Lyapunov function in the sense of **Assumption 3**.

• Weak regularity assumptions are needed on the coefficients β and σ to obtain existence and uniqueness results for the viscosity solution.

THEOREM

For every probability distribution P_0 on D, there exists one and only one stochastic process X solution of the martingale problem for (L, P_0) with $\mathcal{D}L = \mathcal{C}_c^2(D)$. X is a strong Markov process with cádlág (right continuous with left end limits) paths in D.

Denoting by X^x the process with $P_0 = \delta_x$, $x \in D$, it holds, for every $T \geq 0$ and every $\left\{\mathcal{F}_t^{X^x}\right\}$-stopping time τ,

$$\sup_{0 \leq t \leq T} \mathbf{E}\left[V\left(X_{t \wedge \tau}^x\right)\right] \leq C_T\left(1 + V(x)\right).$$

... Moreover,

▶ **Proposition.** *For every* $g \in \mathcal{C}^\infty([0,T] \times D)$ *such that satisfies*

$$|g(t,x)|\varphi(|g(t,x)|) \leq C\,(1 + V(x))$$

and for every compact set $K \subseteq D$*,*

$$g(t \wedge \tau_K, X^x_{t\wedge\tau_K}) - \int_0^{t\wedge\tau_K} (\partial_s g + Lg)\,(s, X^x_s)ds, \quad 0 \leq t \leq T,$$

is an $\{\mathcal{F}_t^{X^x}\}$*-martingale, where*

$$\tau_K = \inf\{t \geq 0 : \;\; X^x_t \notin K \text{ or } X^x_{t-} \notin K\}$$

THEOREM 2. *For every $x \in D$, let X^x the process related to $P_0 = \delta_x$. Then, for every $t \in [0,T]$,*

$$\mathbf{E}\left[\left|\phi(X^x_{T-t})e^{-\int_0^{T-t} c(X^x_r)dr} - \int_0^{T-t} f(t+s, X^x_s)e^{-\int_0^s c(X^x_r)dr} ds\right|\right] < +\infty.$$

$$u(t,x) = \mathbf{E}\left[\phi(X^x_{T-t})e^{-\int_0^{T-t} c(X^x_r)dr} - \int_0^{T-t} f(t+s, X^x_s)e^{-\int_0^s c(X^x_r)dr} ds\right]$$

is continuous on $[0,T] \times D$ and is uniformly continuous on compact subsets of D, uniformly for $t \in [0,T]$. Moreover u is a viscosity solution of $(\mathbf{P})$, satisfying

$$(\mathbf{G}) \qquad |u(t,x)|\varphi(|u(t,x)|) \leq C_T\,(1 + V(x))\,.$$

THEOREM 3. $\exists$ *only one viscosity solution of $(\mathbf{P})$ satisfying $(\mathbf{G})$.*

▶ The results are presented in a paper accepted for publication in **Finance and Stochastics** (2010).

$\Longrightarrow$ **Existence of the stochastic process X:**

- **Step 1:** For every bounded open set D', $\overline{D'} \subseteq D$, we consider the localized operator

$$L' = A' + J',$$

$$A' : \mathcal{C}_c^2(\mathbf{R}^d) \to \mathcal{C}_c(\mathbf{R}^d), \quad A'g(x) = \chi'(x)\left[\nabla g(x)b(x) + \tfrac{1}{2}\mathrm{tr}\left(\nabla^2 g(x)a(x)\right)\right],$$

$$J' : \mathcal{C}_c^2(\mathbf{R}^d) \to \mathcal{C}_c(\mathbf{R}), \quad J'g(x) = \chi'(x)\int_D [g(z) - g(x)]\, m(x, dz),$$

with $\chi' \in \mathcal{C}_c^\infty(D)$, $\chi'(x) = 1$ for $x \in D'$, $0 \le \chi' \le 1$.

  – Since a may be degenerate, standard theorems on Lévy generators do not apply to L' .
  – Under our assumptions the martingale problem for A' is well posed but this is not enough to ensure that the martingale problem for L' is well posed (Theorem 10.3 in Ethier, Kurtz, (1986)).

- $a_{i,j} \in \mathcal{C}^2(D) \Longrightarrow$
  the closure of A' generates a Feller semigroup on $\widehat{\mathcal{C}}(\mathbf{R}^d)$ (the closure of $\mathcal{C}_c(\mathbf{R}^d)$ in $\mathcal{C}_b(\mathbf{R}^d)$) $\Longrightarrow$
  the closure of L' generates a Feller semigroup on $\widehat{\mathcal{C}}(\mathbf{R}^d) \Longrightarrow$
  the martingale problem for L' is well posed

- D starshaped $\Longrightarrow$ D can be made into a complete, separable metric space

$\Longrightarrow$ the stopped martingale problem for (L, P_0, D') has one and only one solution for every probability distribution P_0 on D (see Ethier and Kurtz, (1986)).

- **Step 2:** (using a technique developed by Has'minski, (1980)) For an increasing sequence of sets $D_n \nearrow D$, letting X^n be the solution of the stopped martingale problem for (L, δ_x, D_n) and

$$\tau_n = \inf \left\{ t \geq 0 : \quad X_t^n \notin D_n \text{ or } X_{t-}^n \notin D_n \right\},$$

  $\mathbf{P}(\tau_n \leq t) \longrightarrow 0 \Longrightarrow$ The martingale problem for (L, δ_x) has a unique solution with càdlàg paths.

- **Step 3:** For any probability distribution P_0 on D and any solution X of the martingale problem for (L, P_0), we get

$$\mathbf{P}(\tau_n \leq t, \ X_0 \in K) \leq \frac{2\, \mathbf{P}(X_0 \in K) \left[\sup_{x \in K} V(x) + Ct \right] \mathrm{e}^{Ct}}{n}$$

  $\Longrightarrow$ The martingale problem for (L, P_0) has one and only one solution with càdlàg paths .

  $\Longrightarrow$ X is a strong Markov process (Theorems 4.2, 4.6 in Ethier and Kurtz, (1986)).

Proof of Theorem 2:

- We use the fact that X^x is the solution of the martingale problem for the operator L on $\mathcal{C}_c^2(D)$, rather than the semigroup property.
- This allows locally Lipschitz coefficients which satisfy general growth conditions, and with jump diffusion processes.
- **Step 1:** We show that $u \in \mathcal{C}([0,T] \times D)$.

  Let $\overline{f}$ be an extension of f to $[0,\infty) \times D$ and $Y_t^x = \mathrm{e}^{-\int_0^t c(X_r^x)dr}$,

$$\phi(X_{T-t_k}^{x_k})Y_{T-t_k}^{x_k} - \int_0^{T-t_k} f(t_k+s, X_s^{x_k})Y_s^{x_k}ds$$

$$\xrightarrow{\mathcal{L}} \phi(X_{T-t}^{x})Y_{T-t}^{x} - \int_0^{T-t} f(t+s, X_s^{x})Y_s^{x}ds.$$

$\Longrightarrow$ The uniform integrability of random variables in the left hand side allows to take expectations.

Step 3: We verify the definition of viscosity solution for u.

$$\mathcal{A}g(x) = \nabla g(x)b(x) + \frac{1}{2}\mathrm{tr}\left(\nabla^2 g(x)a(x)\right), \quad Jg(x) = \int_D [g(z) - g(x)]\, m(x, dz).$$

Definition (Viscosity Solution). A function $u \in \mathcal{USC}([0,T] \times D)$ (resp. $u \in \mathcal{LSC}([0,T] \times D)$) such that $\int_D u(t,x)\ m(x,dz) < \infty$ for all $(t,x) \in [0,T] \times D$ is a **viscosity subsolution** (resp. **supersolution**) of (**P**) if

i) for every $(t,x) \in (0,T) \times D$ and any test function g for $\mathcal{P}^{2,+}u$ at (t,x) (resp. $\mathcal{P}^{2,-}u$), it holds:

$$\partial_t g(t,x) + \mathcal{A}g(t,x) + Ju(t,x) - c(x)u(t,x) \geq f(t,x) \quad (resp. \leq),$$

ii) $u(T,x) \leq \phi(x)$, (resp., $\geq$) for all $x \in D$.

A function u that is both a viscosity subsolution and a viscosity supersolution of (**P**) is a **viscosity solution**.

. . . Notice that, for every $\bar{t} \in [0, T)$, the strong Markov property of X^x implies that

$$u(\bar{t} + t, X_t^x) e^{-\int_0^t c(X_r^x) dr} - \int_0^t f(s, X_s^x) e^{-\int_0^s c(X_r^x) dr} ds, \quad 0 \leq t \leq T - \bar{t},$$

is an $\{\mathcal{F}_t^{X^x}\}$-martingale.

Suppose that for a **good test function** for $\mathcal{P}^{2,+} u\ (\bar{t}, \overline{x})$, with $|g(t,x)|\varphi(|g(t,x)|)| \leq C(1 + V(x))$, it holds

$$\partial_t g(\bar{t}, \overline{x}) + \mathcal{A} g(\bar{t}, \overline{x}) + J u(\bar{t}, \overline{x}) - c(\bar{t}, \overline{x}) g(\bar{t}, \overline{x}) - f(\bar{t}, \overline{x}) < 0.$$

Therefore $\exists\ \delta > 0$ such that for $(t, x) \in [\bar{t}, \bar{t} + \delta) \times B_\delta(\overline{x})$,

$$\partial_t g(t, x) + L g(t, x) - c(t, x) g(t, x) - f(t, x) < 0.$$

Set $\overline{\tau} = \inf\left\{t \geq 0 : \ \ X_t^{\overline{x}} \notin B_\delta(\overline{x}) \text{ or } X_{t-}^{\overline{x}} \notin B_\delta(\overline{x})\right\}$, then

$$g(\overline{t} + t \wedge \overline{\tau}, X_{t\wedge\overline{\tau}}^{\overline{x}}) - \int_0^{t\wedge\overline{\tau}} (\partial_s g + Lg)\,(\overline{t} + s, X_s^{\overline{x}})ds, \qquad 0 \leq t \leq T - \overline{t}$$

is an $\left\{\mathcal{F}_t^{X^{\overline{x}}}\right\}$-martingale. Hence

$$u(\overline{t}, \overline{x}) \leq \mathbf{E}\Bigg[g(\overline{t} + \overline{\tau}\wedge\delta, X_{\overline{\tau}\wedge\delta}^{\overline{x}})\mathrm{e}^{-\int_0^{\overline{\tau}\wedge\delta} c(X_r^{\overline{x}})dr} - \int_0^{\overline{\tau}\wedge\delta} f(\overline{t} + s, X_s^{\overline{x}})\mathrm{e}^{-\int_0^{\overline{\tau}\wedge\delta} c(X_r^{\overline{x}})dr} ds\Bigg] =$$

$$g(\overline{t}, \overline{x}) + \mathbf{E}\left[\int_0^{\overline{\tau}\wedge\delta} (\partial_s g + Lg - cg - f)\,(\overline{t} + s, X_s^{\overline{x}})\mathrm{e}^{-\int_0^{\overline{\tau}\wedge\delta} c(X_r^{\overline{x}})dr} ds\right]$$

$< u(\overline{t}, \overline{x}). \qquad \Longrightarrow \qquad$ **Contradiction!**

- The growth condition(**G**) follows from the assumptions of ϕ and f.

Proof of Theorem 3 follows from a **Comparison Principle:** *Let* $\underline{u}$ *and* $\overline{u}$ *be respectively a viscosity sub-/super-solution of* $(\mathbf{P})$*, both satisfying* $(\mathbf{G})$*. Then* $\underline{u}(t,x) \leq \overline{u}(t,x), \ \forall\, (t,x) \in [0,T] \times D$*.*

For every $\beta > 0$, we consider the function

$$w_\beta(t,x,y) = \underline{u}(t,x) - \overline{u}(t,y) - \beta[1 + V(x) + V(y)].$$

The assertion follows proving that the *usc* envelope of

$$\vartheta_\beta(t) = \lim_{r \to 0^+} \sup \Big\{ \max(w_\beta(t,x,y), 0) : \, x, y \in D, \, |x - y| < r \Big\},$$

is a viscosity subsolution of $\left(\vartheta_\beta^*\right)'(t) = 0$ in $(0,T)$.

► Theorems proved in [**Ishii, Kobayasi (1994)**] yield $\vartheta_\beta(t) = 0$, for all t, $\beta > 0$.

Regularity in the Degenerate Purely Differential Case

- **Further Assumptions:** $\langle a(x)\eta, \eta\rangle > 0 \;\forall\, \eta \in \mathbb{R}^n, \;\; x \in D$, f is locally Hölder continuous on $(0,T)\times D$, c is locally Hölder continuous on D.
- **Theorem** The viscosity solution u to $(\mathbf{P})$ is a classical solution.
- **Lemma** Consider a cylinder $H = (t_1, t_2) \times Q$ with $\overline{H} \subset (0,T)\times D$ and let g be a continuous function on the parabolic boundary $\partial_p H$. Then there is a unique classical solution to the Dirichlet problem

$$\begin{cases} \partial_t u + \mathcal{A}u - cu = f, & \text{in } H \\ u = g, & \text{on } \partial_p H \end{cases}$$

Precisely, we have $u \in \mathcal{C}(\overline{H}) \cap \mathcal{C}^{1,2}(H)$.

- The proof follows from classical results in Friedman (1975).

... for the sake of simplicity assume $f = 0$. For a fixed point $(\bar{t}, \overline{x}) \in (0, T) \times D$ and $r > 0$, let us consider the problem

$$\begin{cases} \partial_t v + \mathcal{A} v - cv = 0, & \text{in } (\bar{t}, \bar{t} + r) \times Q \\ v = u, & \text{on } \partial_p[(\bar{t}, \bar{t} + r) \times Q] \end{cases}$$

where $\overline{x} \in Q$, $\overline{Q} \subset D$. Then the unique solution v is classical. Since

$$u(\bar{t} + t \wedge \tau, X^{\overline{x}}_{t\wedge\tau}) e^{-\int_0^{t\wedge\tau} c(X^{\overline{x}}_\lambda) d\lambda}$$

is an $\{\mathcal{F}^X_t\}$-martingale for every $\{\mathcal{F}^X_t\}$-stopping time τ, we consider the first exit time τ of $(X^{\overline{x}}_{s-\bar{t}})_{s \geq \bar{t}}$ from Q and, by applying Ito's formula to $v(t, X^{\overline{x}}_{t-\bar{t}}) e^{-\int_0^{t-\bar{t}} c(X^{\overline{x}}_\lambda) d\lambda}$, it is easy to see that

$$u(\bar{t}, \overline{x}) = \mathbf{E}\left[v(\tau, X^{\overline{x}}_{\tau-\bar{t}}) e^{-\int_0^\tau c(X^{\overline{x}}_\lambda) d\lambda} \right] = v(\bar{t}, \overline{x}).$$

$\Longrightarrow$ **we deduce the regularity of the viscosity solution** u

Regularity: a Class of Strongly Degenerate PIDE's

$\exists\, p \in \mathbf{R}^d$, s.t. $\sigma(x)^T p = 0$ in D. Up to a linear change of coordinates,

$$\sigma(x) = \begin{bmatrix} \sigma^1(x) \\ 0 \end{bmatrix}, \quad \sigma^1(x) \in \mathbf{R}^{d_1 \times d}, \;\; d_1 < d.$$

$$D = D^1 \times D^2, \quad b(x) = \begin{bmatrix} b^1(x^1, x^2) \\ 0 \end{bmatrix}, \quad \mu(x,z) = \delta_{x^1}\left(dz^1\right) \times \mu^2\left(x^2, dz^2\right),$$

i.e. the stochastic process X_t is given by $\left(X_t^1, X_t^2\right)$, where X_t^1 satisfies

$$X_s^1 = x^1 + \int_t^s b^1(X_\lambda^1, X_\lambda^2)d\tau + \int_t^s \sigma^1(X_\lambda^1, X_\lambda^2)dW_\lambda,$$

and X_t^2 is a pure jump Markov process with bounded intensity the evolution of which does not depend on the value of X_t^1.

Since $D = D^1 \times D^2$ and X^2 is a pure jump process with bounded intensity, **Assumption 3** can be replaced by the following

▶**Assumption 3.1:** $\exists\, V^1 \in \mathcal{C}^2(D^1)$ such that

$$V^1(x^1) \geq 0, \ \ \lim_{x^1 \in D^1,\, |x^1| \to +\infty} V^1(x^1) = +\infty,$$

$$\lim_{x^1 \to x_0^1} V^1(x^1) = +\infty \quad \forall\, x_0^1 \in \partial D^1.$$

$$LV^1(x) \leq C\left(1 + V^1(x^1)\right), \quad \forall\, x = (x^1, x^2) \in D.$$

▶**Further Assumptions:**

- $\sigma^1(x)\left(\sigma^1(x)\right)^\top$ is positive definite in D **(but not uniformly)**;
- c is locally Hölder continuous;
- $t \in (0,T) \mapsto f(t,x)$ is Hölder-$\frac{1}{2}$, uniformly w.r.to $x \in D$.

THEOREM 4. Under these assumptions, **the (unique) viscosity solution of (P)**, $u(t, x^1, x^2)$, belongs to $C^{1,2,0}\left((0,T) \times D^1 \times D^2\right)$.

Conclusions

- We have proved existence and uniqueness of the viscosity solution to the valuation equation for a general jump-diffusion model with locally Lipschitz continuous coefficients.

- Our assumptions allow the diffusion matrix $a = \sigma\sigma^\top$ to be singular and both σ and the drift b to loose Lipschitz continuity at the boundary of the state space D.

. . . Our results apply to: **Asian option pricing in stochastic volatility models as well**, **Path-dependent volatility models**, **Jump-diffusion stochastic volatility models**