Operators of Kolmogorov type and parabolic operators associated with non-commuting vector fields: obstacle problems and boundary behaviour

Kaj Nyström

Umeå University, Sweden

Non-divergence form parabolic operators

Let $1 \le m \le N$ and consider $(x,t) \in \mathbb{R}^{N+1}$.

- Uniformly elliptic operators

$$L = \sum_{i,j=1}^{N} a_{ij}(x,t)\partial_{x_i x_j} - \partial_t$$

- Operators of Kolmogorov type

$$L = \sum_{i,j=1}^{m} a_{ij}(x,t)\partial_{x_i x_j} + \sum_{i=1}^{m} a_i(x,t)\partial_{x_i} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t$$

- Operators associated with non-commuting vector fields

$$L = \sum_{i,j=1}^{m} a_{ij}(x,t) X_i X_j - \partial_t$$

Outline of the talk

1. Operators of Kolmogorov type
2. Obstacle problems for operators of Kolmogorov type
    - (Optimal) interior regularity and regularity at the initial state
    - Smooth and non-smooth obstacles
3. Operators associated with non-commuting vector fields
4. Boundary behaviour (for non-negative solutions vanishing on the boundary) for operators associated with non-commuting vector fields
    - Backward Harnack inequality
    - Boundary Hölder continuity of quotients
    - Doubling property of the L-parabolic measure
5. Work in progress

Collaborators

- Operators of Kolmogorov type
  - Marie Frentz
  - Chiara Cinti
  - Andrea Pascucci
  - Sergio Polidoro
- Operators associated with non-commuting vector fields
  - Marie Frentz
  - Nicola Garofalo
  - Elin Götmark
  - Isidro Munive

Operators of Kolmogorov type - an example

1. $1 \leq m$, $N = 2m$.

$$L = X_1^2 + .. + X_m^2 + Y,\ X_i = \partial_{x_i},\ Y = x_1\partial_{x_{m+1}} + .. + x_m\partial_{x_{2m}} - \partial_t$$

2. Generators for the following system of SDEs

$$dX_1 = dW_1,\ ..,\ dX_m = dW_m,\ dX_{m+1} = X_1 dt,\ ..,\ dX_{2m} = X_m dt.$$

3. Hypoellipticity: $[X_i, Y] = \partial_{x_{m+i}}$.
4. Dilation at (0,0,0): $(x', x'', t) := (x_1, .., x_m, x_{m+1}, .., x_{2m}, t)$, $x' \to \lambda x'$, $x'' \to \lambda^3 x''$, $t \to \lambda^2 t$.
5. Group law: L is invariant w.r.t left translations of the Lie group $(\mathbb{R}^{N+1}, \circ)$,

$$(x', x'', t) \circ (\xi', \xi'', \tau) = (\xi' + x', \xi'' + x'' - \tau x', \tau + t).$$

Operators of Kolmogorov type

$z = (x, t) \in \mathbb{R}^{N+1}$, $1 \le m \le N$. General equations of the form

$$L = \sum_{i,j=1}^{m} a_{ij}(z)\partial_{x_i x_j} + \sum_{i=1}^{m} a_i(z)\partial_{x_i} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t. \qquad (1)$$

[H.1] $A_0(z) = \{a_{ij}(z)\}_{i,j=1,\dots,m}$ is symmetric and uniformly positive definite in $\mathbb{R}^m$:

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}(z)\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^m,\ z \in \mathbb{R}^{N+1}.$$

[H.2] The constant coefficient (frozen) operator

$$\mathcal{K} = \sum_{i,j=1}^{m} a_{ij}(z_0)\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t = \sum_{i=1}^{m} X_i^2 + Y \qquad (2)$$

is hypoelliptic.

[H.2] is equivalent to the following structural assumption on B : there exists a basis for $\mathbb{R}^N$ such that the matrix B has the form

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_\kappa \\ * & * & * & \cdots & * \end{pmatrix} \tag{3}$$

where B_j is a $m_{j-1} \times m_j$ matrix of rank m_j for $j \in \{1, \dots, \kappa\}$, $1 \le m_\kappa \le \ldots \le m_1 \le m_0 = m$ and $m + m_1 + \ldots + m_\kappa = N$, while $*$ represents arbitrary matrices with constant entries.

[H.3] a_{ij} and a_i belong to the space $C_K^{0,\alpha}(\mathbb{R}^{N+1})$ of Hölder continuous functions for some $\alpha \in]0,1]$.

Operators of Kolmogorov type

Dilations: Based on the structure of B one can introduce a family of dilations $(\delta_\lambda)_{\lambda>0}$ on $\mathbb{R}^{N+1}$ defined by

$$\delta_\lambda := \mathrm{diag}(\lambda I_m, \lambda^3 I_{m_1}, \dots, \lambda^{2\kappa+1} I_{m_\kappa}, \lambda^2), \tag{4}$$

where I_k, $k \in \mathbb{N}$, is the k-dimensional unit matrix.
Group law: relevant Lie group related to the operator $\mathcal{K}$ in (2)

$$(x,t)\circ(\xi,\tau) = (\xi+\exp(-\tau B^T)x, t+\tau), \; (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{5}$$

[H.4] The operator $\mathcal{K}$ in (2) is δ_λ-homogeneous of degree two, i.e.

$$\mathcal{K} \circ \delta_\lambda = \lambda^2(\delta_\lambda \circ \mathcal{K}), \; \forall \lambda > 0.$$

Remark: **[H.4]** is satisfied if (and only if) all the blocks denoted by $*$ in (3) are null.

Obstacle problem for operators of Kolmogorov type

$\Omega \subset \mathbb{R}^{N+1}$: an open subset.
$g, f, \psi : \bar{\Omega} \to \mathbb{R}$, continuous and bounded on $\bar{\Omega}$ and $g \geq \psi$ on $\bar{\Omega}$.

$$\begin{cases} \max\{Lu(x,t) - f(x,t), \psi(x,t) - u(x,t)\} = 0, & \text{in } \Omega, \\ u(x,t) = g(x,t), & \text{on } \partial_P \Omega. \end{cases} \tag{6}$$

1. $u \in S^1_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a strong solution to problem (6) if the differential inequality is satisfied a.e. in Ω and the boundary datum is attained at any point of $\partial_P \Omega$.
2. Existence and uniqueness of a strong solution to (6) have been proved by DiFrancesco, Pascucci, Polidoro (2008).
3. By Sobolev embedding: $u \in C^{1,\alpha}_{K,\text{loc}}(\Omega)$.
4. Applications to American options: geometric average Asian options, option pricing in the stochastic volatility model of Hobson-Rogers (1998).

Main results - optimal interior regularity

Theorem 1

Assume **H1**-**H4**. *Let* $\alpha \in]0,1]$ *and let* Ω, Ω' *be domains of* $\mathbb{R}^{N+1}$, $\Omega' \subset\subset \Omega$. *Let u be a solution to problem* (6). *Then*

i) *if* $\psi \in C_K^{0,\alpha}(\Omega)$ *then* $u \in C_K^{0,\alpha}(\Omega')$, $\|u\|_{C_K^{0,\alpha}(\Omega')}$ *bounded by*

$$c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|\psi\|_{C_K^{0,\alpha}(\Omega)}\right);$$

ii) *if* $\psi \in C_K^{1,\alpha}(\Omega)$ *then* $u \in C_K^{1,\alpha}(\Omega')$, $\|u\|_{C_K^{0,\alpha}(\Omega')}$ *bounded by*

$$c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|\psi\|_{C_K^{1,\alpha}(\Omega)}\right);$$

iii) *if* $\psi \in C_K^{2,\alpha}(\Omega)$ *then* $u \in S^\infty(\Omega')$, $\|u\|_{S^\infty(\Omega')}$ *bounded by*

$$c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|\psi\|_{C_K^{2,\alpha}(\Omega)}\right).$$

Main results - optimal regularity at the initial state

Theorem 2

Assume **H1**-**H4**, $\alpha \in]0,1]$, Ω, Ω' *as in Theorem 1. u solution to problem* (6) *in* Ω_{t_0}. $\Omega_{t_0}(\Omega'_{t_0}) := \Omega(\Omega') \cap \{t > t_0\}$, $t_0 \in \mathbb{R}$. *Then*

i) *if* $g, \psi \in C_K^{0,\alpha}(\Omega_{t_0})$ *then* $u \in C_K^{0,\alpha}(\Omega'_{t_0})$, $\|u\|_{C_K^{0,\alpha}(\Omega'_{t_0})}$ *bounded*

$$\text{by } c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|g\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|\psi\|_{C_K^{0,\alpha}(\Omega_{t_0})}\right);$$

ii) *if* $g, \psi \in C_K^{1,\alpha}(\Omega_{t_0})$ *then* $u \in C_K^{1,\alpha}(\Omega'_{t_0})$, $\|u\|_{C_K^{1,\alpha}(\Omega'_{t_0})}$ *bounded*

$$\text{by } c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|g\|_{C_K^{1,\alpha}(\Omega_{t_0})}, \|\psi\|_{C_K^{1,\alpha}(\Omega_{t_0})}\right);$$

iii) *if* $g, \psi \in C_K^{2,\alpha}(\Omega_{t_0})$ *then* $u \in S^{\infty}(\Omega'_{t_0})$, $\|u\|_{S^{\infty}(\Omega'_{t_0})}$ *bounded*

$$\text{by } c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|g\|_{C_K^{2,\alpha}(\Omega_{t_0})}, \|\psi\|_{C_K^{2,\alpha}(\Omega_{t_0})}\right).$$

Regularity at the initial state - the proof

$$S_k^+(u-F) = \sup_{Q^+_{2^{-k}}} |u-F|, \qquad (7)$$

where $F = P_n^{(0,0)} g$, $n \in \{0,1,2\}$, $\gamma \in \{\alpha, 1+\alpha, 2\}$.
Key estimate: $\exists\, c > 0$ such that $S^+_{k+1}(u-F)$ is bounded, for all $k \in \mathbb{N}$, by

$$\max\left(c\, 2^{-(k+1)\gamma}, \frac{S_k^+(u-F)}{2^\gamma}, \frac{S^+_{k-1}(u-F)}{2^{2\gamma}}, \dots, \frac{S^+_0(u-F)}{2^{(k+1)\gamma}} \right). \qquad (8)$$

Implication: by dilation, translation, Taylor's formula, (8) it follows that if $(u,f,g,\psi) \in \mathcal{P}_n(L, Q^+_R(x_0,t_0), \alpha, M_1, M_2, M_3, M_4)$, then $\exists\, c = c(L, \alpha, M_1, M_2, M_3, M_4)$,

$$\sup_{Q_r^+(x_0,t_0)} |u-g| \le c r^\gamma, \ r \in]0,R[.$$

Assume that (8) is false and $F = P_0^{(0,0)} g = 0$: $\forall\, j \in \mathbb{N}$, $\exists$ a positive integer k_j, $(u_j, f_j, g_j, \psi_j) \in \mathcal{P}_0(L, Q^+, \alpha, M_1, M_2, M_3, M_4)$, such that

$$S^+_{k_j+1}(u_j) > \max\left(\frac{j\,(C_\alpha + M_3)}{2^{(k_j+1)\alpha}}, \frac{S^+_{k_j}(u_j)}{2^\alpha}, \frac{S^+_{k_j-1}(u_j)}{2^{2\alpha}}, \dots, \frac{S^+_0(u_j)}{2^{(k_j+1)\alpha}} \right).$$

$\exists (x_j, t_j) \in \overline{Q^+_{2^{-k_j-1}}}$, $|u_j(x_j, t_j)| = S^+_{k_j+1}(u_j)$ for every $j \geq 1$.

Let $(\tilde{x}_j, \tilde{t}_j) = \delta_{2^{k_j}}((x_j, t_j))$ and define $\tilde{u}_j : Q^+_{2^{k_j}} \longrightarrow \mathbb{R}$ as

$$\tilde{u}_j(x, t) = \frac{u_j(\delta_{2^{-k_j}}(x, t))}{S^+_{k_j+1}(u_j)}. \tag{9}$$

Note that $(\tilde{x}_j, \tilde{t}_j)$ belongs to the closure of $Q^+_{1/2}$ and

$$\tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1. \tag{10}$$

Regularity at the initial state - the proof in case $n = 0$

Let $\tilde{L}_j = L_{2^{-k_j}}$. Then,

$$\begin{cases} \max\{\tilde{L}_j\tilde{u}_j - \tilde{f}_j, \tilde{\psi}_j - \tilde{u}_j\} = 0, & \text{in } Q^+_{2^{k_j}}, \\ \tilde{u}_j = \tilde{g}_j, & \text{on } \partial_P Q^+_{2^{k_j}}. \end{cases}$$

and

$$(\tilde{u}_j, \tilde{f}_j, \tilde{u}_j, \tilde{\psi}_j) \in \mathcal{P}_0(\tilde{L}_j, Q^+_{2^l}, \alpha, \tilde{M}^j_1, \tilde{M}^j_2, \tilde{M}^j_3, \tilde{M}^j_4),$$

for some $\tilde{M}^j_1, \tilde{M}^j_2, \tilde{M}^j_3, \tilde{M}^j_4$.

$$\tilde{M}^j_1 \leq 2^{(l+1)\alpha}, \lim_{j\to\infty} \tilde{M}^j_2 = \lim_{j\to\infty} m_j = 0, \tag{11}$$

where

$$m_j = \max\left\{ \|\tilde{g}_j\|_{L^\infty(Q^+_{2^l})}, \sup_{Q^+_{2^l}} \tilde{\psi}_j \right\}. \tag{12}$$

Final step: we construct a barrier $\tilde{v}_j$ such that

$$\tilde{u}_j \le \tilde{v}_j \text{ in } Q^+_{2^l}, \qquad (13)$$

and we prove that

$$\sup_{Q^+} \tilde{v}_j \le \frac{1}{2} \text{ for any } j \ge j_0,$$

which contradicts (10). This completes the proof for $n = 0$.

Crucial point: to prove that $m_j \to 0$ as $j \to \infty$.

Operators associated with non-commuting vector

Let $1 \le m \le N$ and consider $z = (x, t) \in \mathbb{R}^{N+1}$.

$$L = \sum_{i,j=1}^{m} a_{ij}(z) X_i X_j - \partial_t. \tag{14}$$

1. $X = \{X_1, ..., X_m\}$: C^∞ vector fields in $\mathbb{R}^N$ satisfying

$$\text{rank Lie}\, [X_1, ..., X_m] \equiv N. \tag{15}$$

2. $A(z) = \{a_{ij}(z)\}_{i,j=1,...,m}$: is symmetric and uniformly positive definite in $\mathbb{R}^m$:

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}(z)\xi_i\xi_j \le \lambda|\xi|^2, \ \forall\, \xi \in \mathbb{R}^m, \ z \in \mathbb{R}^{N+1}. \tag{16}$$

3. $\exists\ C > 0$, and $\sigma \in (0, 1)$, such that for $(x, t), \ (y, s) \in \mathbb{R}^{N+1}$,

$$|a_{ij}(x, t) - a_{ij}(y, s)| \le C d_p(x, t, y, s)^\sigma, \ i, j \in \{1, .., m\}. \tag{17}$$

Boundary behaviour for operators associated with non-commuting vector - Geometry+Dirichlet problem

1. $\Omega \subset \mathbb{R}^N$ bounded domain, $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{N+1}$, $T > 0$ fixed.
2. Ω: NTA domain (non-tangentially accessible domain), with parameters M, r_0, defined w.r.t $X = \{X_1, ..., X_m\}$.
3. If Ω NTA, all points on the parabolic boundary (of Ω_T)
$$\partial_p\Omega_T = S_T \cup (\Omega \times \{0\}), \quad S_T = \partial\Omega \times (0, T),$$
are regular for the Dirichlet problem for L in (14).
4. For any $f \in C(\partial_p\Omega_T)$, $\exists$ a unique $u = u_f^{\Omega_T} \in C(\overline{\Omega}_T)$ to
$$Lu = 0 \text{ in } \Omega_T, \quad u = f \text{ on } \partial_p\Omega_T. \tag{18}$$
5. $\exists$ a unique probability measure $d\omega^{(x,t)}$ on $\partial_p\Omega_T$
$$u(x, t) = \int_{\partial_p\Omega_T} f(y, s) d\omega^{(x,t)}(y, s). \tag{19}$$

Boundary behaviour for operators associated with non-commuting vector - additional notation

1. $B_d(x,r) = \{y \in \mathbb{R}^N : \ d(x,y) < r\}$.
2. For $(x,t) \in \mathbb{R}^{N+1}$ and $r > 0$:
   $C_r^-(x,t) = B_d(x,r) \times (t - r^2, t)$,
   $C_r(x,t) = B_d(x,r) \times (t - r^2, t + r^2)$,
   $\Delta(x,t,r) = S_T \cap C_r(x,t)$.
3. Ω NTA domain with parameters M and r_0: for any $x_0 \in \partial\Omega$, $0 < r < r_0$, $\exists$ a point $A_r(x_0) \in \Omega$, such that

$$M^{-1} r < d(x_0, A_r(x_0)) < r, \ \text{ and } \ d(A_r(x_0), \partial\Omega) \geq M^{-1} r.$$

4. We let $A_r(x_0, t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$.

Theorem 3 [Backward Harnack inequality]

Let u be a non-negative solution of $Lu = 0$ in Ω_T vanishing continuously on S_T. Let $0 < \delta \ll \sqrt{T}$, $(x_0, t_0) \in S_T$, $\delta^2 \leq t_0 \leq T - \delta^2$, $r < \min\{r_0/2, \sqrt{(T - t_0 - \delta^2)/4}, \sqrt{(t_0 - \delta^2)/4}\}$. Then, $\exists\ c = c(L, M, r_0, diam(\Omega), T, \delta)$, $1 \leq c < \infty$, such that if $(x, t) \in \Omega_T \cap C_{r/4}(x_0, t_0)$ then $u(x, t) \leq cu(A_r(x_0, t_0))$.

Theorem 4 [Boundary Hölder continuity of quotients]

Let u, v be non-negative solutions of $Lu = 0$ in Ω_T. $(x_0, t_0) \in S_T$, $r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}$. If u, v vanish continuously on $\Delta(x_0, t_0, 2r)$, then the quotient v/u is Hölder continuous on the closure of $\Omega_T \cap C_r^-(x_0, t_0)$.

Main results

Theorem 5 [Doubling property of the L-parabolic measure]

Let $K \geq 100$ be a fixed constant. Let $(x_0, t_0) \in S_T$, and suppose that $r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}$. Then there exists $c = c(L, M, K, r_0)$, $1 \leq c < \infty$, such that for every $(x, t) \in \Omega_T$, with $d(x_0, x) \leq K|t - t_0|^{1/2}$, $t - t_0 \geq 16r^2$, one has

$$\omega^{(x,t)}(\Delta(x_0, t_0, 2r)) \leq c\omega^{(x,t)}(\Delta(x_0, t_0, r)).$$

Remark: If $m = N$ and $\{X_1, ..., X_m\} = \{\partial_{x_1}, ..., \partial_{x_n}\}$, the proof of Theorem 3-5 culminated with the celebrated papers of Fabes, Safonov and Yuan (1997,1999).

Brief discussion of the proofs of Theorem 3-5

1. Our approach is modeled on the ideas developed by Fabes, Safonov and Yuan.
2. Our proofs use mainly elementary principles: comparison principles, solvability of the Dirichlet problem, interior regularity theory, the (interior) Harnack inequality, Hölder continuity type estimates, decay estimates at the lateral boundary, estimates for the Cauchy problem and fundamental solutions.
3. We heavily use the recent results of Bramanti, Brandolini, Lanconelli and Uguzzoni concerning the (interior) Harnack inequality, the Cauchy problem and the existence and Gaussian estimates for fundamental solutions for the non-divergence form operators L defined in (14).
4. When the matrix $A(x,t) = \{a_{ij}(x,t)\}$ in (14) has entries which are just bounded and measurable, then most of the results we use are presently not known.

Caffarelli's program for obstacle problems

1. Optimal regularity of the solution
2. Non-degeneracy of the solution
3. Qualitative properties of the free boundary
4. Regularity of the free boundary: 'the FB is Lipschitz'
5. Regularity of the free boundary: 'the FB is $C^{1,\alpha}$'
6. Higher regularity

Work in progress

1. Boundary behaviour for operators of Kolmogorov type
2. Regularity of the free boundary in the obstacle problem for operators of Kolmogorov type
3. Obstacle problems (existence, regularity theory) for parabolic operators associated with non-commuting vector

A Carleson-type estimate in Lip(1,1/2)-domains for non-negative solutions to Kolmogorov operators

$x = \left(x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}\right)$, $x^{(0)} \in \mathbb{R}^m$, $x^{(j)} \in \mathbb{R}^{m_j}$, $j \in \{1, \dots, \kappa\}$. Let $x_{I}^{(0)} = x_j^{(0)}$ for some $j \in \{1, ..., m\}$, $x_{II}^{(0)}$ the remaining coordinates. We say that $f : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}$ is a Lip(1,1/2) function with constant M, if $f(0,0) = 0$ and if

$$|f(x_{II}^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) - f(\xi_{II}^{(0)}, \xi^{(1)}, \dots, \xi^{(\kappa)}, \tau)| \leq$$
$$M\left(|x_{II}^{(0)} - \xi_{II}^{(0)}| + |x^{(1)} - \xi^{(1)}| + ... + |x^{(\kappa)} - \xi^{(\kappa)}| + |t - \tau|^{1/2}\right)$$

$$Q_{\rho_1,\rho_2,\rho_3} := \{|x_{II,i_0}^{(0)}| < \rho_1,\ |x_{i_1}^{(1)}| < \rho_2^3, ..., |x_{i_\kappa}^{(\kappa)}| < \rho_2^{2\kappa+1},\ |t| < \rho_3\}.$$

A Carleson-type estimate in Lip(1,1/2)-domains for non-negative solutions to Kolmogorov operators

$$
\begin{aligned}
\Omega_{f,\rho_1,\rho_2,\rho_3} &= \{(x,t) : (x_{II}^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) \in Q_{\rho_1,\rho_2,\rho_3}, \\
&\qquad f(x_{II}^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) < x_I^{(0)} < 16M\rho_1\}, \\
\Delta_{f,\rho_1,\rho_2,\rho_3} &= \{(x,t) : (x_{II}^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) \in Q_{\rho_1,\rho_2,\rho_3}, \\
&\qquad x_I^{(0)} = f(x_{II}^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t)\}.
\end{aligned}
$$

$(\tilde{x}, \tilde{t})$: specific point of reference with $\tilde{x}^{(0)} = 2Me_I^{(0)}$, $\tilde{t} = 1$. We let $A_1^+(0,0) = (\tilde{x}, \tilde{t})$, $A_r^+(0,0) = \delta_r(\tilde{x}, \tilde{t})$ whenever $r > 0$. Given an arbitrary point (x_0, t_0):

$$
\begin{aligned}
\Omega_{f,\rho_1,\rho_2,\rho_3}(x_0,t_0) &= (x_0,t_0) \circ \Omega_{f,\rho_1,\rho_2,\rho_3}, \\
\Delta_{f,\rho_1,\rho_2,\rho_3}(x_0,t_0) &= (x_0,t_0) \circ \Delta_{f,\rho_1,\rho_2,\rho_3}, \\
A_r^+(x_0,t_0) &= (x_0,t_0) \circ A_r^+(0,0).
\end{aligned}
$$

A Carleson-type estimate in Lip(1,1/2)-domains for non-negative solutions to Kolmogorov operators

Theorem 6 [A Carleson-type estimate]

Let f a Lip(1,1/2) function with constant M. Then there exist $1 > r_0 > 0$, $r_0 = r_0(L, N, M)$, $A \geq 4$, $A = A(L, N, M)$, and a point $(\tilde{x}, \tilde{t})$, with $\tilde{x}^{(0)} = 2Me_l^{(0)}$, $\tilde{t} = 1$, such that the following is true. Let $(x_0, t_0) \in \mathbb{R}^{N+1}$, $0 < r < r_0$, assume that u is a non-negative solution to $Lu = 0$ in $\Omega_{f,4r,Ar,4r}(x_0, t_0)$ and that u vanishes continuously on $\Delta_{f,4r,Ar,4r}(x_0, t_0)$. Then there exists a constant $c = c(N, M, A, L)$, $1 \leq c < \infty$, such that

$$u(x, t) \leq cu(A_r^+(x_0, t_0))$$

whenever $(x, t) \in \Omega_{f,r/c,r/c,r/c}(x_0, t_0)$.

Remark: This result generalizes a previous result of S. Salsa (1981) valid for uniformly elliptic-parabolic operators.

Papers

- (with M.Frentz, A.Pascucci and S.Polidoro) Optimal Regularity in the Obstacle Problem for Kolmogorov Operators related to American Asian Options, to appear in Mathematische Annalen.
- (with A. Pascucci and S. Polidoro) Regularity near the Initial State in the Obstacle Problem for a class of Hypoelliptic Ultraparabolic Operators, to appear in Journal of Differential Equations.
- (with C.Cinti and S.Polidoro) A note on Harnack inequalities and propagation sets for a class of hypoelliptic operators, to appear in Potential Analysis.
- (with M.Frentz) Adaptive Stochastic Weak Approximation of Degenerate Parabolic Equations of Kolmogorov type, to appear in Journal of Computational and Applied Math.
- (with C.Cinti and S.Polidoro) A boundary estimate for non-negative solutions to Kolmogorov operators in non-divergence form, submitted.

Papers

- (with C.Cinti and S.Polidoro) A Carleson-type estimate in Lip(1,1/2)-domains for non-negative solutions to Kolmogorov operators, in preparation.
- (with M. Frentz, N. Garofalo, Elin Götmark and I. Munive) Non-divergence form parabolic equations associated with non-commuting vector fields: Boundary behavior of non-negative solutions, submitted.
- (with M. Frentz and Elin Götmark) The Obstacle Problem for Parabolic Non-divergence Operators of Hörmander type, manuscript.
- (with M. Frentz) Regularity in the Obstacle Problem for Parabolic Non-divergence Operators of Hörmander type, manuscript.
- (with M. Frentz and Elin Götmark) Regularity for the Obstacle Problem for Parabolic Non-divergence Operators of Hörmander type, in preparation.