Random noise propagating through a chain of ODEs

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Model description

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Degenerate diffusion

$$\begin{cases} dX_t^1 = F_1(t, X_t^1, \cdots, X_t^n)dt + \sigma(t, \mathbf{X}_t)dW_t, \\ dX_t^2 = F_2(t, X_t^1, \cdots, X_t^n)dt, \\ dX_t^3 = F_3(t, X_t^2, \cdots, X_t^n)dt, \\ \dots \\ dX_t^n = F_n(t, X_t^{n-1}, X_t^n)dt. \end{cases} \tag{1}$$

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- $\mathbf{X}_t = (X_t^1, \cdots, X_t^n) \in \mathbb{R}^{nd}$.

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Microscopic model of heat diffusion (see Heckmann *et al.*)

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Question

Existence of the density and associated controls.

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- $\exists \Lambda \geq 1, \forall t \geq 0, \ \xi \in \mathbb{R}^d, \ \mathbf{x} \in \mathbb{R}^{nd}, \ \Lambda |\xi|^2 \geq \langle a(t,\mathbf{x})\xi, \xi \rangle \geq \Lambda^{-1}|\xi|^2$

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- $F_1, \cdots, F_n, \sigma$ are κ Lipschitz continuous.
  For $2 \leq i \leq n,\ (x_{i-1} \mapsto F_i(t, x_{i-1}, \cdots, x_n)) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and $D_{x_{i-1}} F_i(t, x_{i-1}, \cdots, x_n)$ is κ-η Hölder continuous in space $\eta > 0$.

Assumptions and main results

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- For $2 \leq i \leq n, \ \exists \mathcal{E}_{i-1} \subset GL_d(\mathbb{R})$ closed and convex s.t.
  $\forall (t, x_{i-1}, \cdots, x_n) \in \mathbb{R}^+ \times \mathbb{R}^{n-i+2}, \ D_{x_{i-1}} F_i(t, x_{i-1}, \cdots, x_n) \in \mathcal{E}_{i-1}$.

Running assumptions

Previous conditions: Assumption **(A)**.

Theorem (Delarue, M. (JFA, 2010))

Under **(A)**, for all $(t,\mathbf{x}) \in \mathbb{R}^{+*} \times \mathbb{R}^{nd}$, $\mathbb{P}_{\mathbf{x}}[\mathbf{X}_t \in d\mathbf{y}] = p(t,\mathbf{x},\mathbf{y})d\mathbf{y}$.
For any $T > 0$, $\exists C_T := C_T(\Lambda, \kappa, (\mathcal{E}_i)_{i\in\{1,\cdots,n-1\}}) \geq 1$ s.t. $\forall t \in (0,T]$,

$$C_T^{-1} t^{-n^2 d/2} \exp\left(-C_T t |\mathbb{T}_t^{-1}(\boldsymbol{\theta}_t(\mathbf{x}) - \mathbf{y})|^2\right) \leq p(t,\mathbf{x},\mathbf{y}) \leq C_T t^{-n^2 d/2} \exp\left(-C_T^{-1} t |\mathbb{T}_t^{-1}(\boldsymbol{\theta}_t(\mathbf{x}) - \mathbf{y})|^2\right),$$

with $\dot{\boldsymbol{\theta}}_t = \mathbf{F}(t,\boldsymbol{\theta}_t)$ deterministic counterpart of the SDE and

$$\mathbb{T}_t = \begin{pmatrix} tI_{d\times d} & 0_{d\times d} & \cdots & 0_{d\times d} \\ 0_{d\times d} & t^2 I_{d\times d} & \cdots & \cdots \\ \cdots & \cdots & \ddots & \cdots \\ 0_{d\times d} & \cdots & 0_{d\times d} & t^n I_{d\times d} \end{pmatrix} \text{ (scale matrix).}$$

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- Structure of the equation allows the **separation of scale**. Namely the i^{th} component has typical scale of order $t^{(2i-1)/2}$.

$\rightsquigarrow$ Diagonal exponent $\frac{d}{2}\sum_{i=1}^{n}(2i-1) = \frac{n^2 d}{2}$.

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- Time inhomogeneous dynamics, general Hörmander assumption, but simple propagation of the noise.

Global metric (that includes transport of the initial condition, map $\boldsymbol{\theta}$).

Kusuoka & Stroock, strong Hörmander assumption: only the noise can degenerate.

Difficulty: metric strongly associated to the nature of points involved.

Assumptions and main results

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- **Gaussian bounds**: techniques for the proof.
$\rightsquigarrow$ Linearize dynamics of $\mathbf{X}_t$ to obtain a Gaussian system $\mathbf{X}_t^G$.
$\rightsquigarrow$ Express (minus) the logarithm of the density in terms of a stochastic control problem (Fleming transform).
$\rightsquigarrow$ Compare the Fleming transform of $J(t, \mathbf{x}) = -\log p(t, T, \mathbf{x}, \mathbf{y})$ and $J^G(t, \mathbf{x}) = -\log p^G(t, T, \mathbf{x}, \mathbf{y})$, $T > 0$, $\mathbf{y} \in \mathbb{R}^{nd}$ fixed.

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Take $d = 1, n = 2$,
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Gaussian case and density estimates

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$$(X_t^1, X_t^2) \sim \mathcal{N}\big((x_0, y_0 + a x_0 t), \mathbf{K}_t\big),\ \mathbf{K}_t = \begin{pmatrix} t & \frac{at^2}{2} \\ \frac{at^2}{2} & \frac{a^2 t^3}{3} \end{pmatrix}.$$

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- Additional singularity due to the degeneracy,
- Different characteristic time scales for X^1, X^2.

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where $\forall t \in [0,T],\ \mathrm{Spec}(\Sigma_t \Sigma_t^*) \subset [\Lambda^{-1}, \Lambda],\ \varphi \in L^2([0,T], \mathbb{R}^d)$ and

$$\mathbf{L}_t = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \alpha_t^1 & 0 & \cdots & \cdots & 0 \\ 0 & \alpha_t^2 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \alpha_t^{n-1} & 0 \end{pmatrix} + \mathbf{U}_t,\ \mathbf{U}_t \text{ upper triangular.}$$

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Put $\mathbf{G}_t = \mathbf{S}_t(\dot{W}), \ \ t \in [0, T]$. The vector $\mathbf{G}_t$ is Gaussian. Denote $\mathbf{K}_t$ its covariance matrix.

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$\mathbf{K}_T$ non degenerate $\iff$ $(\mathbf{S}_t)_{t\in[0,T]}$ controllable, i.e. $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^{nd}, \ \exists \varphi \in L^2([0, T], \mathbb{R}^d), \ \mathbf{S}_0(\varphi) = \mathbf{x}, \mathbf{S}_T(\varphi) = \mathbf{y}$.
Off diagonal bound given by the cost (Kalman's theorem):

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$$\begin{aligned} I^0_{\rm linear}(T,\mathbf{x},\mathbf{y}) &= \inf\{\int_0^T |\varphi_t|^2 dt,\ \mathbf{S}_0(\varphi) = \mathbf{x},\ \mathbf{S}_0(\varphi) = \mathbf{y}\} \\ &= \langle \mathbf{R}(T,0)\mathbf{x} - \mathbf{y}, \mathbf{K}_T^{-1}(\mathbf{R}(T,0)\mathbf{x} - \mathbf{y})\rangle. \end{aligned}$$

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$$\partial_t \mathbf{R}(t, t_0) = \mathbf{L}_t \mathbf{R}(t, t_0), \ \mathbf{R}(t_0, t_0) = I_{nd}.$$

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Using the resolvant:

$$\begin{aligned} \mathbf{G}_T &= \mathbf{R}(T,0)\mathbf{x} + \int_0^T \mathbf{R}(T,s) B\Sigma_s dW_s, \\ \mathbf{K}_T &= \int_0^T \mathbf{R}(T,s) B(\Sigma_s \Sigma_s^*) B^* [\mathbf{R}(T,s)]^* ds. \end{aligned}$$

$\mathbf{K}_T$ is the Gram matrix of the controllability probem.

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Density estimate

Under **(A-3)**, for $t > 0,\ \ \mathbf{G}_t$, with $\mathbf{G}_0 = \mathbf{x} \in \mathbb{R}^{nd}$, has a density $q(t, \mathbf{x}, \mathbf{y}),\ \ \mathbf{y} \in \mathbb{R}^{nd}$.

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Under **(A-3)**, for $t > 0$, $\mathbf{G}_t$, with $\mathbf{G}_0 = \mathbf{x} \in \mathbb{R}^{nd}$, has a density $q(t,\mathbf{x},\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^{nd}$. Also, $\exists C, \forall t \in (0,T]$,

$$C^{-1} t^{-n^2 d/2} \exp\left(-Ct|\mathbb{T}_t^{-1}(\mathbf{R}(t,0)(\mathbf{x}) - \mathbf{y})|^2\right) \le q(t,\mathbf{x},\mathbf{y})$$
$$\le Ct^{-n^2 d/2} \exp\left(-C^{-1}t|\mathbb{T}_t^{-1}(\mathbf{R}(t,0)(\mathbf{x}) - \mathbf{y})|^2\right).$$

Stochastic control formulation of the density

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Density satisfies Fokker Planck equation

$$\partial_t p(t,T,\mathbf{x},\mathbf{y}) + \mathcal{L}_t p(t,T,\mathbf{x},\mathbf{y}) = 0,\ 0 \le t < T,\ \mathbf{x},\mathbf{y} \in \mathbb{R}^{nd},$$
$$p(T,T,\mathbf{x},\mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x}),\ \mathbf{x},\mathbf{y} \in \mathbb{R}^{nd},$$

$\mathcal{L}_t = (1/2)\mathrm{Tr}(a(t,\mathbf{x})D^2_{x^1}) + F_1(t,\mathbf{x})D_{x^1} + \sum_{i=2}^n F_i(t,\mathbf{x}^{i-1,n})D_{x^i}.$

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Define $u_\varepsilon(t, \mathbf{x}) := \mathbb{E}_{t,\mathbf{x}}[\eta_\varepsilon(\mathbf{X}_{T-\varepsilon})], (t, \mathbf{x}) \in [0, T-\varepsilon] \times \mathbb{R}^{nd}, \eta_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow}$ $\delta_{\mathbf{y}_0},\ \mathbf{y}_0 \in \mathbb{R}^{nd}$ fixed. Then
$u_\varepsilon(0, \mathbf{x}) = \mathbb{E}_{0,\mathbf{x}}[\eta_\varepsilon(\mathbf{X}_{T-\varepsilon})] \underset{\varepsilon \to 0}{\longrightarrow} p(T, \mathbf{x}, \mathbf{y}_0)$.

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$$
\begin{aligned}
(\partial_t + \mathcal{L}_t)u_\varepsilon(t,\mathbf{x}) &= 0,\ 0 \le t < T-\varepsilon,\ \mathbf{x} \in \mathbb{R}^{nd},\\
u_\varepsilon(T-\varepsilon,\mathbf{x}) &= \eta_\varepsilon(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^{nd}.
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Define $u_\varepsilon(t,\mathbf{x}) := \mathbb{E}_{t,\mathbf{x}}[\eta_\varepsilon(\mathbf{X}_{T-\varepsilon})], (t,\mathbf{x}) \in [0,T-\varepsilon]\times\mathbb{R}^{nd}, \eta_\varepsilon \underset{\varepsilon\to 0}{\to} \delta_{\mathbf{y}_0},\ \mathbf{y}_0\in\mathbb{R}^{nd}$ fixed. Then
$u_\varepsilon(0,\mathbf{x}) = \mathbb{E}_{0,\mathbf{x}}[\eta_\varepsilon(\mathbf{X}_{T-\varepsilon})] \underset{\varepsilon\to 0}{\to} p(T,\mathbf{x},\mathbf{y}_0)$.

$$\begin{aligned}(\partial_t+\mathcal{L}_t)u_\varepsilon(t,\mathbf{x}) &= 0,\ 0\le t< T-\varepsilon,\ \mathbf{x}\in\mathbb{R}^{nd},\\ u_\varepsilon(T-\varepsilon,\mathbf{x}) &= \eta_\varepsilon(\mathbf{x}),\ \mathbf{x}\in\mathbb{R}^{nd}.\end{aligned}$$

Fleming transform:
$J_\varepsilon(t,\mathbf{x}) = -\ln(u_\varepsilon(t,x)),\ (t,x)\in[0,T-\varepsilon]\times\mathbb{R}^{nd}$.
Associated nonlinear PDE: $0\le t<T-\varepsilon,\ \mathbf{x}\in\mathbb{R}^{nd}$,

$$(\partial_t+\mathcal{L}_t)J_\varepsilon(t,\mathbf{x}) - \frac{1}{2}\langle a^{-1}(t,\mathbf{x})D_{x_1}J_\varepsilon(t,\mathbf{x}), D_{x_1}J_\varepsilon(t,\mathbf{x})\rangle = 0,$$
$$J_\varepsilon(T-\varepsilon,\mathbf{x}) = -\log(\eta_\varepsilon(\mathbf{x})),\ \mathbf{x}\in\mathbb{R}^{nd}.$$

Stochastic control formulation of the density

Link with stochastic control: Fleming transform

Fleming transform:
$J_\varepsilon(t, \mathbf{x}) = -\ln(u_\varepsilon(t, x)),\ (t, x) \in [0, T - \varepsilon] \times \mathbb{R}^{nd}$.
Associated nonlinear PDE: $0 \le t < T - \varepsilon,\ \mathbf{x} \in \mathbb{R}^{nd}$,

$$(\partial_t + \mathcal{L}_t) J_\varepsilon(t, \mathbf{x}) - \frac{1}{2}\langle a^{-1}(t, \mathbf{x}) D_{x_1} J_\varepsilon(t, \mathbf{x}), D_{x_1} J_\varepsilon(t, \mathbf{x})\rangle = 0,$$
$$J_\varepsilon(T - \varepsilon, \mathbf{x}) = -\log(\eta_\varepsilon(\mathbf{x})),\ \mathbf{x} \in \mathbb{R}^{nd}.$$

Simple quadratic optimization

$$(\partial_t + \mathcal{L}_t) J_\varepsilon(t, \mathbf{x}) + \inf_{v \in \mathbb{R}^d} \{\langle v, D_{x_1} J_\varepsilon(t, \mathbf{x})\rangle + \frac{1}{2}\langle a^{-1}(t, \mathbf{x}) v, v\rangle\} = 0,$$
$$v^* := v^*(t, \mathbf{x}) = -a(t, \mathbf{x}) D_{x_1} J_\varepsilon(t, \mathbf{x}).$$

Stochastic control formulation of the density

Control formulation for the density

$$d\chi_t = [\mathbf{F}(t,\chi_t) + Bv_t]dt + \sigma(t,\chi_t)dW_t, \ \chi_0 = \mathbf{x}.$$

Stochastic control formulation of the density

Control formulation for the density

$$d\chi_t = [\mathbf{F}(t,\chi_t) + Bv_t]dt + \sigma(t,\chi_t)dW_t,\ \chi_0 = \mathbf{x}.$$

$$J_\varepsilon(0,\mathbf{x}) = \inf_{v\in\mathcal{P}(T-\varepsilon,\mathbf{x})} \mathbb{E}_{0,\mathbf{x}}\left[\frac{1}{2}\int_0^{T-\varepsilon}\langle a^{-1}(t,\chi_t)v_t, v_t\rangle dt - \ln(\eta_\varepsilon(\chi_{T-\varepsilon}))\right].$$

Pathwise, Itô's formula yields:

$$J_\varepsilon(0,\mathbf{x}) = -\ln(\eta_\varepsilon(\chi_{T-\varepsilon})) + \frac{1}{2}\int_0^{T-\varepsilon}\langle a^{-1}(t,\chi_t)v_t, v_t\rangle dt$$
$$-\frac{1}{2}\int_0^{T-\varepsilon}|\sigma^{-1}(t,\chi_t)[v_t^* - v_t]|^2 dt + \int_0^{T-\varepsilon}\langle\sigma^{-1}(t,\chi_t)v_t^*, dW_t\rangle,$$
$$v_t^* = -a(t,\chi_t)D_{x_1}J_\varepsilon(t,\chi_t).$$

Key steps of the proof

What's next?

- **Majoration of J_ε $\Rightarrow$Minoration of p. Free choice of v_t!**
- Compare the previous expansion of J_ε to a similar expression associated to a Gaussian density $\rightsquigarrow$ Linearization, stability.
- Around what? Deterministic control of the non linear deterministic system $\dot{\phi}_t = \mathbf{F}(t, \phi_t) + B\varphi_t$ whose energy is dominated by the density bounds $I(T, \mathbf{x}, \mathbf{y}) = \int_0^T |\varphi_t|^2 dt \le CT|\mathbb{T}_T^{-1}(\theta(\mathbf{x}) - \mathbf{y})|^2$
- Splitting of the diagonal estimate and the off diagonal one.

Key steps of the proof

Stability estimates main steps

- Take ϕ as above for fixed $\mathbf{x}_0, \mathbf{y}_0$ and set $v_t = v_t^0 + \varphi_t$ (splitting diagonal-off diagonal).

Key steps of the proof

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- Take ϕ as above for fixed $\mathbf{x}_0, \mathbf{y}_0$ and set $v_t = v_t^0 + \varphi_t$ (splitting diagonal-off diagonal).
- Write:

$$
\begin{aligned}
J_\varepsilon(0,\mathbf{x}_0) \le -\ln[\eta_\varepsilon(\chi_{T-\varepsilon})] &+ \frac{1}{2}\int_0^{T-\varepsilon} \langle a^{-1}(t,\phi_t)\varphi_t,\varphi_t\rangle dt \\
&+\frac{1}{2}\int_0^{T-\varepsilon} \langle a(t,\phi_t)^{-1} v_t^0, v_t^0\rangle dt + C\int_0^{T-\varepsilon} |\varphi_t||v_t^0| dt \\
&+C\int_0^{T-\varepsilon} |\chi_t-\phi_t|\big[|\varphi_t|^2+|v_t^0|^2\big]dt \\
-\frac{1}{2}\int_0^{T-\varepsilon} |\sigma^{-1}(t,\chi_t)[v_t^*-v_t]|^2 dt &+ \int_0^{T-\varepsilon}\langle \sigma^{-1}(t,\chi_t)v_t^*, dW_t\rangle.
\end{aligned}
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& + C\int_0^{T-\varepsilon} |\chi_t - \phi_t|\big[|\varphi_t|^2 + |v_t^0|^2\big] dt \\
& - \frac{1}{2}\int_0^{T-\varepsilon} |\sigma^{-1}(t,\chi_t)[v_t^* - v_t]|^2 dt + \int_0^{T-\varepsilon} \langle \sigma^{-1}(t,\chi_t) v_t^*, dW_t\rangle.
\end{aligned}
$$

$$
d\mathbf{G}_t = \mathbf{L}_t\mathbf{G}_t dt + B\Sigma_t dW_t,\ \mathbf{G}_0 = 0,\ \mathbf{L}_t = D\mathbf{F}(t,\phi_t),\ \Sigma_t = \sigma(t,\phi_t).
$$

Key steps of the proof

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$$J_\varepsilon(0,\mathbf{x}_0) \le CI(T,\mathbf{x}_0,\mathbf{y}_0) - \ln(q(T,0,0)) + R_{T-\varepsilon}.$$

Upper bound for the density

Parametrix technique

Frozen density:

$$d\tilde{\mathbf{X}}_t = \left[\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y})) + D\mathbf{F}(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))\left(\tilde{\mathbf{X}}_t - \boldsymbol{\theta}_{t,T}(\mathbf{y})\right)\right]dt \\ + B\sigma(t, \boldsymbol{\theta}_{t,T}(\mathbf{y}))dW_t, \quad 0 \le t \le T, \tag{1}$$

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Parametrix expansion (Mc Kean and Singer (JDG, 1967)):

$$\begin{aligned} p(T,\mathbf{x},\mathbf{y}) &= \tilde{p}(0,T,\mathbf{x},\mathbf{y}) + \int_0^T \int_{\mathbb{R}^{nd}} p(t,\mathbf{x},\mathbf{z}) H(t,T,\mathbf{z},\mathbf{y}) dt d\mathbf{z}. \\ H(t,T,\mathbf{x},\mathbf{y}) &= \big[\mathcal{L}_{t,\mathbf{x}} - \tilde{\mathcal{L}}_{t,\mathbf{x}}^{T,\mathbf{y}}\big]\big(\tilde{p}^{T,\mathbf{y}}(t,T,\mathbf{x},\mathbf{y})\big). \end{aligned}$$

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H smoothing kernel compatible with the metric:

$$\begin{aligned} \big|H(t, T, \mathbf{x}, \mathbf{y})\big| &\le C(T-t)^{-1+\frac{\eta}{2}} g_{C,T-t}\big(\mathbf{x} - \boldsymbol{\theta}_{t,T}(\mathbf{y})\big). \\ g_{a,t}(\mathbf{z}) &= t^{-n^2\frac{d}{2}} \exp(-a^{-1} t |\mathbb{T}_t^{-1}\mathbf{z}|^2), \quad a, t > 0, \ \mathbf{z} \in \mathbb{R}^{nd}. \end{aligned}$$

Upper bound (parametrix continued)

Problem. An induction yields:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^{nd}} \tilde{p}(0,t,\mathbf{x},\mathbf{z})|H^{\otimes k}(t,1,\mathbf{z},\mathbf{y})|d\mathbf{z}dt \\
&\leq C(k)\int_0^T \int_{\mathbb{R}^{nd}} (T-t)^{k\frac{\eta}{2}-1} g_{C(k),t}\big(\boldsymbol{\theta}_t(\mathbf{x})-\mathbf{z}\big) \\
&\times g_{C(k),T-t}\big(\mathbf{z}-\boldsymbol{\theta}_{t,T}(\mathbf{y})\big)d\mathbf{z}dt \\
&\leq C(k)\int_0^T (T-t)^{k\frac{\eta}{2}-1} g_{C(k),T}\big(\boldsymbol{\theta}_{t,0}(\mathbf{x})-\boldsymbol{\theta}_{t,T}(\mathbf{y})\big)dt \\
&\leq C(k+1)T^{k\frac{\eta}{2}} g_{C(k+1),T}\big(\boldsymbol{\theta}_T(\mathbf{x})-\mathbf{y}\big).
\end{aligned}$$

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$$\times\, g_{C(k),T-t}\big(\mathbf{z}-\boldsymbol{\theta}_{t,T}(\mathbf{y})\big)d\mathbf{z}dt$$
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$\rightsquigarrow$ Finite parametrix iteration needed.

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$\rightsquigarrow$ Finite parametrix iteration needed.
Tool: Fleming transform once again.

Upper bound (parametrix continued)

Problem. An induction yields:

$$\int_0^T \int_{\mathbb{R}^{nd}} \tilde{p}(0,t,\mathbf{x},\mathbf{z})|H^{\otimes k}(t,1,\mathbf{z},\mathbf{y})|d\mathbf{z}dt$$

$$\leq C(k) \int_0^T \int_{\mathbb{R}^{nd}} (T-t)^{k\frac{\eta}{2}-1} g_{C(k),t}\big(\boldsymbol{\theta}_t(\mathbf{x})-\mathbf{z}\big)$$

$$\times\, g_{C(k),T-t}\big(\mathbf{z}-\boldsymbol{\theta}_{t,T}(\mathbf{y})\big)d\mathbf{z}dt$$

$$\leq C(k) \int_0^T (T-t)^{k\frac{\eta}{2}-1} g_{C(k),T}\big(\boldsymbol{\theta}_{t,0}(\mathbf{x})-\boldsymbol{\theta}_{t,T}(\mathbf{y})\big)dt$$

$$\leq C(k+1)T^{k\frac{\eta}{2}} g_{C(k+1),T}\big(\boldsymbol{\theta}_T(\mathbf{x})-\mathbf{y}\big).$$

If F linear no truncation needed, semi-group property preserved (see Konakov, M., Molchanov (AIHP, 2010?)).

Upper bound (final step)

Proposition

Let $a > 0$ be given. There exists a constant $C(a) > 0$, only depending on a and (**A**)*, such that, for all $t \in [0, T)$ and* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{nd}$,

$$\int_{\mathbb{R}^{nd}} p(t, \mathbf{x}, \mathbf{z})(T - t)^{\frac{n^2 d}{2}} g_{a,T-t}\big(\mathbf{z} - \boldsymbol{\theta}_{t,T}(\mathbf{y})\big) d\mathbf{z} \leq C(a) g_{C(a),T}\big(\boldsymbol{\theta}_T(\mathbf{x}) - \mathbf{y}\big).$$

This allows to truncate the parametrix series.

Alternative approach to lower bound

From the parametrix representation

$$
\begin{aligned}
p(s,t,\mathbf{x},\mathbf{x}') \geq \tilde{p}(s,t,\mathbf{x},\mathbf{x}') \\
&- \sum_{k=1}^{N} \int_s^t \int_{\mathbb{R}^{nd}} \tilde{p}(s,u,\mathbf{x},\mathbf{y}) |H^{\otimes k}(u,t,\mathbf{y},\mathbf{x}')| d\mathbf{y} du \\
&- \int_s^t \int_{\mathbb{R}^{nd}} p(s,u,\mathbf{x},\mathbf{y}) |H^{\otimes (N+1)}(u,t,\mathbf{y},\mathbf{x}')| d\mathbf{y} du.
\end{aligned}
$$

Alternative approach to lower bound

From the parametrix representation

$$p(s,t,\mathbf{x},\mathbf{x}') \geq \tilde{p}(s,t,\mathbf{x},\mathbf{x}') - \sum_{k=1}^{N} \int_s^t \int_{\mathbb{R}^{nd}} \tilde{p}(s,u,\mathbf{x},\mathbf{y}) |H^{\otimes k}(u,t,\mathbf{y},\mathbf{x}')| d\mathbf{y} du - \int_s^t \int_{\mathbb{R}^{nd}} p(s,u,\mathbf{x},\mathbf{y}) |H^{\otimes (N+1)}(u,t,\mathbf{y},\mathbf{x}')| d\mathbf{y} du.$$

Hence,

$$p(s,t,\mathbf{x},\mathbf{x}') \geq \frac{C^{-1}}{(t-s)^{n^2 d/2}} \exp(-C(t-s)|\mathbb{T}_{t-s}^{-1}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{x}')|^2) - C \frac{(t-s)^{\eta/2}}{(t-s)^{n^2 d/2}} \exp(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{x}')|^2),$$

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& - C\frac{(t-s)^{\eta/2}}{(t-s)^{n^2 d/2}} \exp(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(\boldsymbol{\theta}_{t,s}(\mathbf{x}) - \mathbf{x}')|^2),
\end{aligned}
$$

Approach of Il'in, Kalashnikov and Oleinik (Uspehi Mat. Nauk, 1962) to lower bound.

Alternative proof for the Lower bound: Chaining

Chaining using the level sets of the energy functional.
Consider the optimal path $(\phi_s)_{s\in[0,T]}$, $\phi_0 = \mathbf{x}$, $\phi_T = \mathbf{x}'$, $(\varphi_s)_{s\in[0,T]}$ denotes the associated optimal control, we define $t_0 = 0$ and, for all $i \geq 1$:

$$
t_i \begin{cases} := \inf\{t \in [t_{i-1}, T] : \int_{t_{i-1}}^{t} |\varphi_s|^2 ds = \frac{I(T,\mathbf{x},\mathbf{x}')}{M_0}\} \wedge (t_{i-1} + \frac{T}{M_0}) \\ \qquad \text{if } t_{i-1} < T(1 - \frac{2}{M_0}) \\ := T \quad \text{if } t_{i-1} \geq T(1 - \frac{2}{M_0}), \end{cases}
$$

where $M_0 := \lceil K d_T^2(\mathbf{x},\mathbf{x}') \rceil \geq 3$,
$d_T(\mathbf{x},\mathbf{x}') := T^{1/2} |\mathbb{T}_T^{-1}(\boldsymbol{\theta}_{T,0}(\mathbf{x}) - \mathbf{x}')|$.
$\exists M_1 \in [M_0/2, M_0/C_1] \cap \mathbb{N}$, s.t. $t_{M_1} = T$ and

$$
\forall i \in \{0, \cdots, M_1 - 1\},\ C_1 \frac{T}{M_0} \leq \varepsilon_i \leq 2\frac{T}{M_0}.
$$

Alternative proof for the Lower bound: Chaining

Introduce now for all $i \in \{0, \cdots, M_1\}$, $\mathbf{y}_i = \phi_{t_i}$ (in particular $\mathbf{y}_0 = \mathbf{x}, \mathbf{y}_{M_1} = \mathbf{x}'$), and for all $i \in \{1, \cdots, M_1 - 1\}$,

$$B_i := \big\{\mathbf{z} \in \mathbb{R}^{nd} : K^{1/2}\rho\big(|\mathbb{T}^{-1}_{K\rho^2}(\boldsymbol{\theta}_{t_i,t_{i-1}}(\mathbf{y}_{i-1}) - \mathbf{z})| + |\mathbb{T}^{-1}_{K\rho^2}(\mathbf{z} - \boldsymbol{\theta}_{t_i,t_{i+1}}(\mathbf{y}_{i+1}))| \le K^{-1/2}\big\},$$

where $\rho := T^{1/2} d_T(\mathbf{x}, \mathbf{x}')/M_0$.

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$$p(T, \mathbf{x}, \mathbf{x}') \ge \int_{\prod_{i=1}^{M_1-1} B_i} \prod_{i=0}^{M_1-1} p(t_i, t_{i+1}, \mathbf{x}_i, \mathbf{x}_{i+1}) d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_{M_1-1}.$$

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Suitable controls yield the bound. Namely:

$$\forall(\mathbf{x}_i, \mathbf{x}_{i+1}) \in B_i \times B_{i+1},\ \varepsilon_i^{1/2}|\mathbb{T}^{-1}_{\varepsilon_i}(\boldsymbol{\theta}_{t_{i+1},t_i}(\mathbf{x}_i) - \mathbf{x}_{i+1})| \le 1,$$
$$\forall i \in \{1, \cdots, M_1 - 1\}, \quad |B_i| \ge C_1 \rho^{n^2 d},$$

Alternative proof for the Lower bound: Chaining

Approach technically different but similar nature than:

- Pascucci and Polidoro (TAMS, 2006): Harnack inequalities .
- Bally and Kohatsu Higa (JFA, 2010): Tubes around "good paths".

Towards uniqueness in law

- Only η-Hölder continuity in space ($\eta > 0$) of the coefficients $\sigma, D_{\mathbf{x}_{i-1}} F_i,\ i \in \{1, ..., n\}$ is needed in our proof that involves regularized coefficient.

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* $\eta > 1/2$: general theory of viscosity solutions (see Ishii, Lions, JDE, 1990) gives the uniqueness in law.

$\rightsquigarrow$ does not exploit the smoothing properties of the heat kernel.

* General η: Schauder estimates or corollary of the Calderón-Zygmund approach to the martingale problem.