Almost monotonicity formulas for elliptic and parabolic operators with variable coefficients

Norayr Matevosyan Arshak Petrosyan

UNIVERSITY OF CAMBRIDGE PURDUE UNIVERSITY

Kolmogorov Equations in Physics and Finance

September 9, 2010

Original Elliptic Monotonicity Formula

Theorem (Alt-Caffarelli-Friedman 1984)

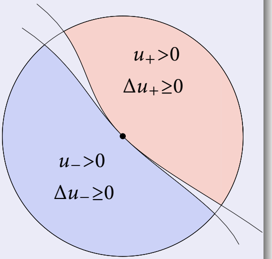
Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad \text{in} \quad B_1$$

then the functional

$$\varphi(r) = \varphi(r, u_+, u_-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_+|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla u_-|^2}{|x|^{n-2}} dx$$

is monotone nondecreasing in $r \in (0,1]$.



Original Elliptic Monotonicity Formula

Theorem (Alt-Caffarelli-Friedman 1984)

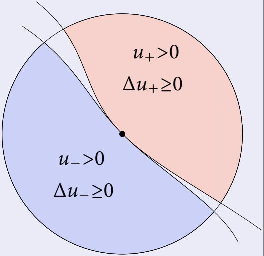
Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad in \quad B_1$$

then the functional

$$\varphi(r) = \varphi(r, u_+, u_-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_+|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla u_-|^2}{|x|^{n-2}} dx$$

is monotone nondecreasing in $r \in (0, 1]$.



- This formula has been of fundamental importance in the regularity theory of free boundaries, especially in problems with two phases.

Original Elliptic Monotonicity Formula

- One of the applications of the monotonicity formula is the ability to produce estimates of the type

$$c_n|\nabla u_+(0)|^2|\nabla u_-(0)|^2 \le \varphi(0+) \le \varphi(1/2) \le C_n\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}$$

Original Elliptic Monotonicity Formula

- One of the applications of the monotonicity formula is the ability to produce estimates of the type

$$c_n|\nabla u_+(0)|^2|\nabla u_-(0)|^2 \le \varphi(0+) \le \varphi(1/2) \le C_n\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}$$

- The proof is based on the following eigenvalue inequality of Friedland-Hayman 1976.

Original Elliptic Monotonicity Formula

- One of the applications of the monotonicity formula is the ability to produce estimates of the type
$$c_n|\nabla u_+(0)|^2|\nabla u_-(0)|^2 \leq \varphi(0+) \leq \varphi(1/2) \leq C_n\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}$$
- The proof is based on the following eigenvalue inequality of Friedland-Hayman 1976.
- For $\Sigma \subset \partial B_1$ define
$$\lambda(\Sigma) = \inf \frac{\int_\Sigma |\nabla_\theta f|^2}{\int_\Sigma f^2}, \quad f|_{\partial\Sigma} = 0$$
Define also $\alpha(\Sigma)$ so that $\lambda(\Sigma) = \alpha(\Sigma)(n-2+\alpha(\Sigma))$.

Original Elliptic Monotonicity Formula

- One of the applications of the monotonicity formula is the ability to produce estimates of the type

$$c_n|\nabla u_+(0)|^2|\nabla u_-(0)|^2 \le \varphi(0+) \le \varphi(1/2) \le C_n\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}$$

- The proof is based on the following eigenvalue inequality of Friedland-Hayman 1976.
- For $\Sigma \subset \partial B_1$ define

$$\lambda(\Sigma) = \inf \frac{\int_\Sigma |\nabla_\theta f|^2}{\int_\Sigma f^2}, \quad f|_{\partial\Sigma} = 0$$

Define also $\alpha(\Sigma)$ so that $\lambda(\Sigma) = \alpha(\Sigma)(n-2+\alpha(\Sigma))$.

Theorem (Friedland-Hayman 1976)

Let $\Sigma_\pm$ be disjoint open sets on ∂B_1. Then

$$\alpha(\Sigma_+) + \alpha(\Sigma_-) \ge 2.$$

Parabolic Monotonicity Formula

Theorem (Caffarelli 1993)

Let $u_\pm(x,s)$ *be two continuous functions in* $S_1 = \mathbb{R}^n \times (-1,0]$

$$u_\pm \geq 0, \quad (\Delta - \partial_s)u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad in \quad S_1$$

then

$$\Phi(r,u_+,u_-) = \frac{1}{r^4} \int_{-r^2}^{0}\int_{\mathbb{R}^n} |\nabla u_+|^2 G(x,-s)\,dx\,ds \int_{-r^2}^{0}\int_{\mathbb{R}^n} |\nabla u_-|^2 G(x,-s)\,dx\,ds$$

is monotone nondecreasing for $r \in (0,1]$.

Parabolic Monotonicity Formula

Theorem (Caffarelli 1993)

Let $u_\pm(x,s)$ be two continuous functions in $S_1 = \mathbb{R}^n \times (-1,0]$

$$u_\pm \geq 0, \quad (\Delta - \partial_s)u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad \textit{in} \quad S_1$$

then

$$\Phi(r,u_+,u_-) = \frac{1}{r^4}\int_{-r^2}^{0}\int_{\mathbb{R}^n}|\nabla u_+|^2 G(x,-s)\,dx\,ds \int_{-r^2}^{0}\int_{\mathbb{R}^n}|\nabla u_-|^2 G(x,-s)\,dx\,ds$$

is monotone nondecreasing for $r \in (0,1]$.

- Note that $u_\pm$ must be defined in a entire strip and we must have a moderate growth at infinity.

Parabolic Monotonicity Formula

- The proof is now based on the eigenvalue inequality in Gaussian space.

Parabolic Monotonicity Formula

- The proof is now based on the eigenvalue inequality in Gaussian space. For $\Omega \subset \mathbb{R}^n$ define

$$\lambda(\Omega) = \inf \frac{\int_\Omega |\nabla f|^2 \, d\nu}{\int_\Omega f^2 \, d\nu}, \qquad d\nu = (2\pi)^{-n/2} e^{-x^2/2} dx.$$

Parabolic Monotonicity Formula

- The proof is now based on the eigenvalue inequality in Gaussian space. For $\Omega \subset \mathbb{R}^n$ define

$$\lambda(\Omega) = \inf \frac{\int_\Omega |\nabla f|^2 \, dv}{\int_\Omega f^2 \, dv}, \qquad dv = (2\pi)^{-n/2} e^{-x^2/2} dx.$$

Theorem (Beckner-Kenig-Pipher)

Let $\Omega_\pm$ be two disjoint open sets in $\mathbb{R}^n$. Then

$$\lambda(\Omega_+) + \lambda(\Omega_-) \geq 2$$

- The proof is reduced to the convexity result of Brascamp-Lieb 1976 for first eigenvalues of $-\Delta + V(x)$ with convex potential V as a function of the domain.

Localized Parabolic Formula

Theorem (Caffarelli 1993)

Let $u_\pm(x,s)$ *be two continuous functions in* $Q_1^- = B_1 \times (-1,0]$ *such that*

$$u_\pm \geq 0, \quad (\Delta - \partial_s)u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad \textit{in} \quad Q_1^-.$$

Let $\psi \in C_0^\infty(B_1)$ *be a cutoff function such that*

$$0 \leq \psi \leq 1, \quad \operatorname{supp}\psi \subset B_{3/4}, \quad \psi|_{B_{1/2}} = 1$$

then $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ *is* **almost monotone** *in a sense that*

$$\Phi(0+) - \Phi(r) \leq Ce^{-c/r^2} \|u_+\|^2_{L^2(Q_1^-)} \|u_-\|^2_{L^2(Q_1^-)}.$$

Generalization: Caffarelli-Kenig Estimate

- Instead of the heat operator $\Delta - \partial_s$ consider now uniformly parabolic

$$\mathcal{L}u = \mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s) \cdot \nabla u + c(x,s)u - \partial_s u$$

Generalization: Caffarelli-Kenig Estimate

- Instead of the heat operator $\Delta - \partial_s$ consider now uniformly parabolic

$$\mathcal{L}u = \mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

- Assume $\mathcal{A}$ to be Dini continuous, b, c uniformly bounded

Generalization: Caffarelli-Kenig Estimate

- Instead of the heat operator $\Delta - \partial_s$ consider now uniformly parabolic

$$\mathcal{L}u = \mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

- Assume $\mathcal{A}$ to be Dini continuous, b, c uniformly bounded

Theorem (Caffarelli-Kenig 1998)

Let $u_\pm(x,s)$ be two continuous functions in Q_1^- such that

$$u_\pm \geq 0, \quad \mathcal{L}u_\pm \geq 0, \quad u_+\cdot u_- = 0 \quad in \quad Q_1^-.$$

Let $\psi \in C_0^\infty(B_1)$ be a cutoff function as before. Then $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ is **almost monotone** *in a sense that we have an estimate*

$$\Phi(r) \leq C_0\left(\|u_+\|^2_{L^2(Q_1^-)} + \|u_-\|^2_{L^2(Q_1^-)}\right)^2, \quad r < r_0.$$

Generalization: Caffarelli-Jerison-Kenig Estimate

Theorem (Caffarelli-Jerison-Kenig 2002)

Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad in \quad B_1$$

then the functional $\varphi(r) = \varphi(r, u_+, u_-)$ satisfies

$$\varphi(r) \leq C_0 \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2, \quad r < r_0.$$

Generalization: Caffarelli-Jerison-Kenig Estimate

Theorem (Caffarelli-Jerison-Kenig 2002)

Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad in \quad B_1$$

then the functional $\varphi(r) = \varphi(r, u_+, u_-)$ satisfies

$$\varphi(r) \leq C_0 \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2, \quad r < r_0.$$

- The proof is based on a sophisticated iteration scheme.

Generalization: Caffarelli-Jerison-Kenig Estimate

Theorem (Caffarelli-Jerison-Kenig 2002)

Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad in \quad B_1$$

then the functional $\varphi(r) = \varphi(r, u_+, u_-)$ satisfies

$$\varphi(r) \leq C_0 \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2, \quad r < r_0.$$

- The proof is based on a sophisticated iteration scheme.
- The difficulties in CJK and CK estimates are of completely different nature

Generalization: Caffarelli-Jerison-Kenig Estimate

Theorem (Caffarelli-Jerison-Kenig 2002)

Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad in \quad B_1$$

then the functional $\varphi(r) = \varphi(r, u_+, u_-)$ satisfies

$$\varphi(r) \leq C_0 \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2, \quad r < r_0.$$

- The proof is based on a sophisticated iteration scheme.
- The difficulties in CJK and CK estimates are of completely different nature
- The proof can be easily generalized to parabolic case (Edquist-Petrosyan 2008).

Almost Monotonicity Formulas

- In CJK and CK estimates there is essentially no monotonicity left

Almost Monotonicity Formulas

- In CJK and CK estimates there is essentially no monotonicity left
- However, we still have an estimate of the type

$$\varphi(0+) \leq C\left(\|u_+\|_{L^2(B_1)}, \|u_-\|_{L^2(B_1)}\right)$$

which is able to produce an estimate

$$|\nabla u_+(0)||\nabla u_-(0)| \leq C\left(\|u_+\|_{L^2(B_1)}, \|u_-\|_{L^2(B_1)}\right).$$

This is crucial in proving the optimal regularity in certain two-phase problems (and not only!)

Almost Monotonicity Formulas

- In CJK and CK estimates there is essentially no monotonicity left
- However, we still have an estimate of the type

$$\varphi(0+) \le C\left(\|u_+\|_{L^2(B_1)}, \|u_-\|_{L^2(B_1)}\right)$$

which is able to produce an estimate

$$|\nabla u_+(0)||\nabla u_-(0)| \le C\left(\|u_+\|_{L^2(B_1)}, \|u_-\|_{L^2(B_1)}\right).$$

This is crucial in proving the optimal regularity in certain two-phase problems (and not only!)

- Under certain growth assumptions on u, such as $|u(x)| \le C|x|^\epsilon$ one can show the existence of $\varphi(0+)$. This is important in classification of blowup solutions.

CJK+CK

- Natural question to ask whether there is a combination of CJK and CK estimates.

CJK+CK

- Natural question to ask whether there is a combination of CJK and CK estimates.
- Namely, do we have an almost monotonicity estimate for $u_\pm$ satisfying

$$u_\pm \geq 0, \quad \mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \text{in} \quad Q_1^-.$$

CJK+CK

- Natural question to ask whether there is a combination of CJK and CK estimates.
- Namely, do we have an almost monotonicity estimate for $u_\pm$ satisfying

$$u_\pm \geq 0, \quad \mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \text{in} \quad Q_1^-.$$

- We will see that the answer is positive when $\mathcal{A}$ is double Dini and b, c are uniformly bounded.

Main Results: Assumptions

- We consider the uniformly parabolic operator

$$\mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

such that

Main Results: Assumptions

- We consider the uniformly parabolic operator

$$\mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

such that

1. $\lambda|\xi|^2 \leq \mathcal{A}(x,s)\xi\cdot\xi \leq \frac{1}{\lambda}|\xi|^2$

Main Results: Assumptions

- We consider the uniformly parabolic operator

$$\mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

such that

1. $\lambda|\xi|^2 \le \mathcal{A}(x,s)\xi\cdot\xi \le \frac{1}{\lambda}|\xi|^2$
2. $\|\mathcal{A}(x,s) - \mathcal{A}(0,0)\| \le \omega\left((|x|^2+s)^{1/2}\right)$ with double Dini ω:

$$\int_0^1 \frac{1}{r}\int_0^r \frac{\omega(\rho)}{\rho}\,d\rho dr = \int_0^1 \frac{\omega(\rho)\log\frac{1}{\rho}}{\rho}\,d\rho < \infty$$

Main Results: Assumptions

- We consider the uniformly parabolic operator

$$\mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

such that

1. $\lambda|\xi|^2 \le \mathcal{A}(x,s)\xi\cdot\xi \le \frac{1}{\lambda}|\xi|^2$
2. $\|\mathcal{A}(x,s) - \mathcal{A}(0,0)\| \le \omega\left((|x|^2+s)^{1/2}\right)$ with double Dini ω:

$$\int_0^1 \frac{1}{r}\int_0^r \frac{\omega(\rho)}{\rho}\,d\rho dr = \int_0^1 \frac{\omega(\rho)\log\frac{1}{\rho}}{\rho}\,d\rho < \infty$$

3. $|b(x,s)| + |c(x,s)| \le \mu$

Main Results: Assumptions

- We consider the uniformly parabolic operator

$$\mathcal{L}_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

  such that

  1. $\lambda|\xi|^2 \leq \mathcal{A}(x,s)\xi\cdot\xi \leq \frac{1}{\lambda}|\xi|^2$
  2. $\|\mathcal{A}(x,s) - \mathcal{A}(0,0)\| \leq \omega\left((|x|^2+s)^{1/2}\right)$ with double Dini ω:

$$\int_0^1 \frac{1}{r}\int_0^r \frac{\omega(\rho)}{\rho}\,d\rho dr = \int_0^1 \frac{\omega(\rho)\log\frac{1}{\rho}}{\rho}\,d\rho < \infty$$

  3. $|b(x,s)| + |c(x,s)| \leq \mu$

- We make similar assumption on the uniformly elliptic operator

$$\ell_{\mathcal{A},b,c}u := \operatorname{div}(\mathcal{A}(x)\nabla u) + b(x)\cdot\nabla u + c(x)u$$

Global Parabolic Formula

Theorem (Matevosyan-Petrosyan)

Let $u_\pm(x,s)$ be two continuous functions in S_1 such that

$$u_\pm \geq 0, \quad \mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \text{in} \quad S_1$$

Assume also that $u_\pm$ have moderate growth at infinity, so that

$$M_\pm^2 := \iint_{S_1} u_\pm(x,s)^2 e^{-x^2/32} dx ds < \infty.$$

Then the functional $\Phi(r) = \Phi(r, u_+, u_-)$ satisfies

$$\Phi(r) \leq C_\omega (1 + M_+^2 + M_-^2)^2, \quad \text{for } 0 < r \leq r_\omega.$$

Localized Parabolic Formula

Theorem (Matevosyan-Petrosyan)

Let $u_\pm(x,s)$ be two continuous functions in Q_1^- such that

$$u_\pm \geq 0, \quad \mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \textit{in} \quad Q_1^-$$

Let also ψ be a cutoff function such that

$$0 \leq \psi \leq 1, \quad \operatorname{supp}\psi \subset B_{3/4}, \quad \psi|_{B_{1/2}} = 1.$$

Then the functional $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ satisfies

$$\Phi(r) \leq C_{\omega,\psi}\left(1 + \|u_+\|^2_{L^2(Q_1^-)} + \|u_-\|^2_{L^2(Q_1^-)}\right)^2, \quad \textit{for } 0 < r \leq r_\omega.$$

Elliptic Formula

Theorem (Matevosyan-Petrosyan)

Let $u_\pm(x)$ be two continuous functions in B_1 such that

$$u_\pm \geq 0, \quad \ell_{\mathcal{A},b,c} u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad in \quad B_1.$$

Then the functional $\varphi(r) = \varphi(r, u_+, u_-)$ satisfies

$$\varphi(r) \leq C_\omega \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2, \quad for\ 0 < r \leq r_\omega.$$

Proof: Key Technical Estimate

- Let $A^{\pm}(r) = \iint_{S_r} |\nabla u|^2 G(x,-s)\,dx\,ds, \quad S_r = \mathbb{R}^n \times (-r^2, 0]$

Proof: Key Technical Estimate

- Let $A^{\pm}(r) = \iint_{S_r} |\nabla u|^2 G(x, -s)\, dx\, ds, \quad S_r = \mathbb{R}^n \times (-r^2, 0]$
- Then $\Phi(r) = r^{-4} A^+(r) A^-(r)$

Proof: Key Technical Estimate

- Let $A^{\pm}(r) = \iint_{S_r} |\nabla u|^2 G(x,-s)\,dx\,ds, \quad S_r = \mathbb{R}^n \times (-r^2, 0]$
- Then $\Phi(r) = r^{-4} A^+(r) A^-(r)$

Proposition (Matevosyan-Petrosyan)

Let $u \geq 0$ *satisfy* $\mathcal{L}_{\mathcal{A},b,0} u \geq -1$ *in* S_1. *Suppose also* $\iint_{S_1} u(x,s)^2 e^{-x^2/32}\,dx\,ds \leq 1$. *Then*

$$(1 - c_n\,\theta(r)) \iint_{S_r} |\nabla u|^2 G(x,-s)\,dx\,ds \leq$$

$$C_0 r^4 + C_n r^2 \left(\int_{\mathbb{R}^n} u(x,-r^2)^2 G(x,r^2)\,dx \right)^{1/2} + \frac{1}{2} \int_{\mathbb{R}^n} u(x,-r^2)^2 G(x,r^2)\,dx$$

for any $0 < r \leq r_\omega$. , *where*

$$\theta(r) = Cr + \omega(r^{1/2}) + \left(\int_0^{r^2} \frac{\omega(\rho^{1/4})^2}{\rho}\,d\rho \right)^{1/2}.$$

Main Technical Part

- $g(r) = \int_0^r \frac{\theta(\rho)}{\rho}\, d\rho$

Main Technical Part

- $g(r) = \int_0^r \frac{\theta(\rho)}{\rho}\, d\rho$
- $\widetilde{A}^{\pm}(r) = e^{c_0 g(r)} A^{\pm}(r), \quad \widetilde{\Phi}(r) = r^{-4} \widetilde{A}^{+}(r) \widetilde{A}^{-}(r)$

Main Technical Part

- $g(r) = \int_0^r \frac{\theta(\rho)}{\rho}\, d\rho$
- $\widetilde{A}^{\pm}(r) = e^{c_0 g(r)} A^{\pm}(r), \quad \widetilde{\Phi}(r) = r^{-4}\widetilde{A}^{+}(r)\widetilde{A}^{-}(r)$

Proposition (Matevosyan-Petrosyan)

Then there exists a universal constant C_0 such that if $\tilde{A}_{\pm}(\rho) \geq C_0 r^4$ for all $\rho \in [\frac{1}{4}r, r]$, $0 < r \leq r_\omega$, then

$$\tilde{\Phi}'(\rho) \geq -C_0 r \left[\frac{1}{\sqrt{\tilde{A}_{+}(\rho)}} + \frac{1}{\sqrt{\tilde{A}_{-}(\rho)}} \right] \tilde{\Phi}(\rho)$$

for all $\rho \in [\frac{1}{4}r, r]$.

Main Technical Part

- $g(r) = \int_0^r \frac{\theta(\rho)}{\rho}\, d\rho$
- $\widetilde{A}^{\pm}(r) = e^{c_0 g(r)} A^{\pm}(r), \quad \widetilde{\Phi}(r) = r^{-4}\widetilde{A}^{+}(r)\widetilde{A}^{-}(r)$

Proposition (Matevosyan-Petrosyan)

Then there exists a universal constant C_0 such that if $\tilde{A}_{\pm}(\rho) \geq C_0 r^4$ for all $\rho \in [\frac{1}{4}r, r]$, $0 < r \leq r_\omega$, then

$$\tilde{\Phi}'(\rho) \geq -C_0 r \left[\frac{1}{\sqrt{\tilde{A}_+(\rho)}} + \frac{1}{\sqrt{\tilde{A}_-(\rho)}} \right] \tilde{\Phi}(\rho)$$

for all $\rho \in [\frac{1}{4}r, r]$.

- We may replace $\tilde{A}_{\pm}$ by $A_{\pm}$ in the above proposition, since the factor $e^{c_0 g(r)}$ is bounded away from 0 and ∞. Yet, we must take the derivative of $\tilde{\Phi}$ to compensate for having $\theta(r)$ in the key estimate of $A^{\pm}(r)$.

Proof: CJK Iteration Scheme for $\mathcal{L} = \Delta - \partial_s$

- Define $A_k^\pm = A^\pm(4^{-k})$, $b_k^\pm = 4^{4k} A_k^\pm$. Then $\Phi(4^{-k}) = 4^{4k} A_k^+ A_k^-$.

Proof: CJK Iteration Scheme for $\mathcal{L} = \Delta - \partial_s$

- Define $A_k^\pm = A^\pm(4^{-k})$, $b_k^\pm = 4^{4k} A_k^\pm$. Then $\Phi(4^{-k}) = 4^{4k} A_k^+ A_k^-$.

Proposition

There exists C_0 such that if $b_k^\pm \geq C_0$ then

$$4^4 A_{k+1}^+ A_{k+1}^- \leq A_k^+ A_k^- (1 + \delta_k) \quad \textit{with} \quad \delta_k = \frac{C_0}{\sqrt{b_k^+}} + \frac{C_0}{\sqrt{b_k^-}}.$$

Proof: CJK Iteration Scheme for $\mathcal{L} = \Delta - \partial_s$

- Define $A_k^\pm = A^\pm(4^{-k})$, $b_k^\pm = 4^{4k} A_k^\pm$. Then $\Phi(4^{-k}) = 4^{4k} A_k^+ A_k^-$.

Proposition

There exists C_0 such that if $b_k^\pm \geq C_0$ then

$$4^4 A_{k+1}^+ A_{k+1}^- \leq A_k^+ A_k^- (1 + \delta_k) \quad \text{with} \quad \delta_k = \frac{C_0}{\sqrt{b_k^+}} + \frac{C_0}{\sqrt{b_k^-}}.$$

Proposition

There exists C_0 such that if $b_k^\pm \geq C_0$ and $4^4 A_{k+1}^+ \geq A_k^+$ then

$$A_{k+1}^- \leq (1 - \epsilon_0) A_k^-.$$

Proof: CJK Iteration Scheme for $\mathcal{L}_{\mathcal{A},b,c}$

Define

- $\widetilde{A}_k^{\pm} = \widetilde{A}^{\pm}(4^{-k}), \widetilde{b}^{\pm} = 4^{4k}\widetilde{A}_k^{\pm}.$

Proof: CJK Iteration Scheme for $\mathcal{L}_{\mathcal{A},b,c}$

Define

- $\widetilde{A}_k^\pm = \widetilde{A}^\pm(4^{-k})$, $\widetilde{b}^\pm = 4^{4k}\widetilde{A}_k^\pm$.

Proposition

$\widetilde{A}_k^\pm$ satisfy the same iterative inequalities as $A_k^\pm$ in the case of $\mathcal{L} = \Delta - \partial_s$.

Proof: Parabolic ⇒ Elliptic

- Add a "dummy" variable s

$$\widetilde{u}_\pm(x,s) = u_\pm(x), \quad (x,s) \in Q_1^-$$

Proof: Parabolic ⇒ Elliptic

- Add a "dummy" variable s

$$\widetilde{u}_\pm(x,s) = u_\pm(x), \quad (x,s) \in Q_1^-$$

- $\widetilde{u}_\pm$ satisfy now conditions of localized parabolic case with

$$\mathcal{L}u = (\ell - \partial_s)u = \operatorname{div}(\mathcal{A}(x)\nabla u) + b(x)\nabla u + c(x)u - \partial_s u.$$

Proof: Parabolic ⇒ Elliptic

- Add a "dummy" variable s

$$\widetilde{u}_\pm(x,s) = u_\pm(x), \quad (x,s) \in Q_1^-$$

- $\widetilde{u}_\pm$ satisfy now conditions of localized parabolic case with

$$\mathcal{L}u = (\ell - \partial_s)u = \operatorname{div}(\mathcal{A}(x)\nabla u) + b(x)\nabla u + c(x)u - \partial_s u.$$

- Fix a cutoff function $\psi \geq 0$ such that $\psi = 1$ on $B_{1/2}$. Note that

$$\int_{B_r} \frac{|\nabla u(x)|}{|x|^{n-2}} dx \leq C_n \iint_{S_r} |\nabla(\psi(x)u(x))|^2 G(x,-s)\,dx\,ds.$$

Proof: Parabolic ⇒ Elliptic

- Add a "dummy" variable s

$$\widetilde{u}_\pm(x,s) = u_\pm(x), \quad (x,s) \in Q_1^-$$

- $\widetilde{u}_\pm$ satisfy now conditions of localized parabolic case with

$$\mathcal{L}u = (\ell - \partial_s)u = \operatorname{div}(\mathcal{A}(x)\nabla u) + b(x)\nabla u + c(x)u - \partial_s u.$$

- Fix a cutoff function $\psi \geq 0$ such that $\psi = 1$ on $B_{1/2}$. Note that

$$\int_{B_r} \frac{|\nabla u(x)|}{|x|^{n-2}} dx \leq C_n \iint_{S_r} |\nabla(\psi(x)u(x))|^2 G(x,-s)\,dx\,ds.$$

- Hence

$$\begin{aligned}
\varphi(r,u_+,u_-) &\leq C_n \Phi(r,\psi\tilde{u}_+,\psi\tilde{u}_-) \\
&\leq C_\omega \left(1 + \|\widetilde{u}_+\|^2_{L^2(Q_1^-)} + \|\widetilde{u}_-\|^2_{L^2(Q_1^-)}\right)^2 \\
&= C_\omega \left(1 + \|u_+\|^2_{L^2(B_1)} + \|\, u_-\|^2_{L^2(B_1)}\right)^2
\end{aligned}$$

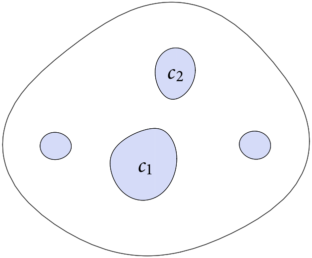
for $r < r_\omega$.

Application: Quasilinear Obstacle-Type Problem

- Let u be a solution of the system in B_1

$$\begin{aligned} \operatorname{div}(a(|\nabla u|^2)\nabla u) &= f(x,u,\nabla u)\chi_\Omega, \\ |\nabla u| &= 0 \quad \text{on } \Omega^c, \end{aligned}$$

where Ω is an apriori unknown open set.



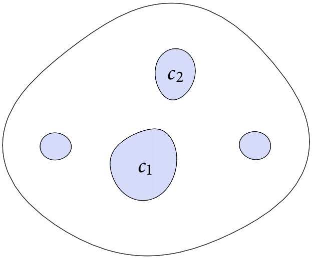
Application: Quasilinear Obstacle-Type Problem

- Let u be a solution of the system in B_1

$$\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x, u, \nabla u)\chi_\Omega,$$
$$|\nabla u| = 0 \quad \text{on } \Omega^c,$$

  where Ω is an apriori unknown open set.

- Problem appears in the description of type II superconductors (Berestycki-Bonnet-Chapman 1994)



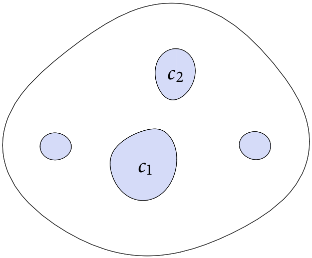
Application: Quasilinear Obstacle-Type Problem

- Let u be a solution of the system in B_1

$$\begin{aligned} \operatorname{div}(a(|\nabla u|^2)\nabla u) &= f(x,u,\nabla u)\chi_\Omega, \\ |\nabla u| = 0 \quad &\text{on } \Omega^c, \end{aligned}$$

  where Ω is an apriori unknown open set.

- Problem appears in the description of type II superconductors (Berestycki-Bonnet-Chapman 1994)
- One-phase problem, however, no assumption is made on the sign of u in Ω



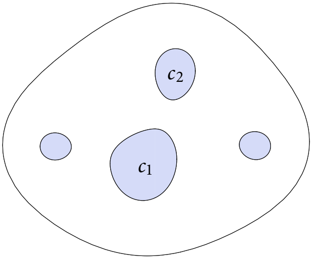
Application: Quasilinear Obstacle-Type Problem

- Let u be a solution of the system in B_1

$$\begin{aligned}\operatorname{div}(a(|\nabla u|^2)\nabla u) &= f(x,u,\nabla u)\chi_\Omega,\\ |\nabla u| = 0 &\quad \text{on } \Omega^c,\end{aligned}$$

  where Ω is an apriori unknown open set.

- Problem appears in the description of type II superconductors (Berestycki-Bonnet-Chapman 1994)
- One-phase problem, however, no assumption is made on the sign of u in Ω
- $\Lambda = \Omega^c$ may break out into different patches Λ_i so that $u = c_i$ on Λ_i



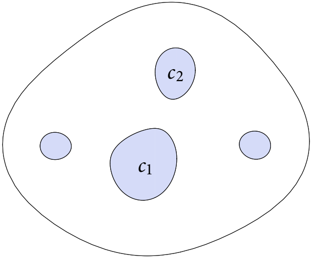
Application: Quasilinear Obstacle-Type Problem

- Let u be a solution of the system in B_1

$$\begin{aligned} \operatorname{div}(a(|\nabla u|^2)\nabla u) &= f(x,u,\nabla u)\chi_\Omega, \\ |\nabla u| &= 0 \quad \text{on } \Omega^c, \end{aligned}$$

  where Ω is an apriori unknown open set.

- Problem appears in the description of type II superconductors (Berestycki-Bonnet-Chapman 1994)
- One-phase problem, however, no assumption is made on the sign of u in Ω
- $\Lambda = \Omega^c$ may break out into different patches Λ_i so that $u = c_i$ on Λ_i



- Similar problem has been studied by Caffarelli-Salazar-Shahgholian 2004

Application: Quasilinear Obstacle-Type Problem

Assumptions

1. $a \in C^{1,\alpha}_{\text{loc}}([0, \infty))$

Application: Quasilinear Obstacle-Type Problem

Assumptions

1. $a \in C^{1,\alpha}_{\text{loc}}([0,\infty))$
2. $a(q), a(q) + 2a'(q)q \in [\lambda_0, 1/\lambda_0]$ for any $q \geq 0$

Application: Quasilinear Obstacle-Type Problem

Assumptions

1. $a \in C^{1,\alpha}_{\mathrm{loc}}([0,\infty))$
2. $a(q), a(q) + 2a'(q)q \in [\lambda_0, 1/\lambda_0]$ for any $q \geq 0$
3. $|f| + |\nabla_x f| + |\partial_z f| + |\nabla_p f| \leq M$ for $(x,z,p) \in D \times \mathbb{R} \times \mathbb{R}^n$.

Application: Quasilinear Obstacle-Type Problem

Assumptions

1. $a \in C^{1,\alpha}_{\mathrm{loc}}([0,\infty))$
2. $a(q), a(q) + 2a'(q)q \in [\lambda_0, 1/\lambda_0]$ for any $q \geq 0$
3. $|f| + |\nabla_x f| + |\partial_z f| + |\nabla_p f| \leq M$ for $(x, z, p) \in D \times \mathbb{R} \times \mathbb{R}^n$.

Theorem (Matevosyan-Petrosyan)

Under conditions above, $u \in C^{1,1}_{\mathrm{loc}}(B_1)$ and

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C\left(C_a, \alpha, n, \lambda_0, M, \|u\|_{L^\infty(B_1)}\right)$$

with $C_a = \|a\|_{C^{1,\alpha}([0,R(n,\lambda_0,M,\|u\|_{L^\infty(B_1)})])}$.

Application: Quasilinear Obstacle-Type Problem

Assumptions

1. $a \in C^{1,\alpha}_{\mathrm{loc}}([0,\infty))$
2. $a(q), a(q) + 2a'(q)q \in [\lambda_0, 1/\lambda_0]$ for any $q \geq 0$
3. $|f| + |\nabla_x f| + |\partial_z f| + |\nabla_p f| \leq M$ for $(x,z,p) \in D \times \mathbb{R} \times \mathbb{R}^n$.

Theorem (Matevosyan-Petrosyan)

Under conditions above, $u \in C^{1,1}_{\mathrm{loc}}(B_1)$ and

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C\left(C_a, \alpha, n, \lambda_0, M, \|u\|_{L^\infty(B_1)}\right)$$

with $C_a = \|a\|_{C^{1,\alpha}([0, R(n,\lambda_0,M,\|u\|_{L^\infty(B_1)})])}$.

- Generalizes a theorem of Shahgholian 2003 for

$$\Delta u = f(x,u)\chi_\Omega, \quad |\nabla u| = 0 \text{ on } \Omega^c.$$

Application: Quasilinear Obstacle-Type Problem

- Connection with the almost monotonicity formulas:

Application: Quasilinear Obstacle-Type Problem

- Connection with the almost monotonicity formulas:

Lemma

For any direction e the functions $w_\pm = (\partial_e u)^\pm = \max\{\pm\partial_e u, 0\}$ *satisfy*

$$w_\pm \geq 0, \quad \operatorname{div}(\mathcal{A}(x)\nabla w_\pm) + b(x)\nabla w_\pm + c(x)w_\pm \geq -M, \quad w_+ \cdot w_- = 0,$$

where

$$\begin{aligned} \mathcal{A}(x) &= a(|\nabla u(x)|^2)I + 2a'(|\nabla u(x)|^2)\nabla u(x) \otimes \nabla u(x), \\ b(x) &= -(\nabla_p f)(x, u(x), \nabla u(x)), \\ c(x) &= -(\partial_z f)(x, u(x), \nabla u(x)). \end{aligned}$$

Application: Quasilinear Obstacle-Type Problem

Idea of the proof (Shahgholian 2003)

- $u \in W^{2,p}$, $p > n$, hence twice differentiable a.e.

Application: Quasilinear Obstacle-Type Problem

Idea of the proof (Shahgholian 2003)

- $u \in W^{2,p}$, $p > n$, hence twice differentiable a.e.
- take $e \perp \nabla u(x_0)$ and apply almost monotonicity formula to $w_\pm = (\partial_e u)^\pm$:

$$|\nabla w(x_0)|^4 \leq C_n \varphi(0+, w^+, w^-) \leq \left(1 + \|w\|^2_{L^2(B_{1/2})}\right)^2,$$

Application: Quasilinear Obstacle-Type Problem

Idea of the proof (Shahgholian 2003)

- $u \in W^{2,p}$, $p > n$, hence twice differentiable a.e.
- take $e \perp \nabla u(x_0)$ and apply almost monotonicity formula to $w_\pm = (\partial_e u)^\pm$:

$$|\nabla w(x_0)|^4 \le C_n \varphi(0+, w^+, w^-) \le \left(1 + \|w\|^2_{L^2(B_{1/2})}\right)^2,$$

- this implies that

$$|\partial_{ee} u(x_0)| \le C, \quad \text{for } e \perp \nabla u(x_0)$$

Application: Quasilinear Obstacle-Type Problem

Idea of the proof (Shahgholian 2003)

- $u \in W^{2,p}$, $p > n$, hence twice differentiable a.e.
- take $e \perp \nabla u(x_0)$ and apply almost monotonicity formula to $w_\pm = (\partial_e u)^\pm$:

$$|\nabla w(x_0)|^4 \leq C_n \varphi(0+, w^+, w^-) \leq \left(1 + \|w\|^2_{L^2(B_{1/2})}\right)^2,$$

- this implies that

$$|\partial_{ee} u(x_0)| \leq C, \quad \text{for } e \perp \nabla u(x_0)$$

- to obtain the estimate in the missing direction $e \parallel \nabla u(x_0)$, we use the equation.

A Variant of the Almost Monotonicity Formula

Theorem (Matevosyan-Petrosyan)

Let $u_\pm$ *satisfy* $u_\pm \geq 0$, $\mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1$, $u_+ \cdot u_- = 0$ *in* S_1, *and*

$$u_\pm(x,s) \leq \sigma((|x|^2 + |s|)^{1/2}) \quad \textit{for } (x,s) \in Q_1^-$$

for a Dini modulus of continuity $\sigma(r)$. *Then* $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ *satisfies*

$$\Phi(r) \leq [1 + \alpha(\rho)]\Phi(\rho) + C_{M,\psi,\sigma,\omega}\alpha(\rho), \quad 0 < r \leq \rho \leq r_\omega,$$

where $\alpha(r) = C_0 \left[r + \sigma(r^{1/2}) + \int_0^r \frac{\sigma(\rho^{1/2})}{\rho} d\rho + \int_0^r \frac{\theta(\rho)}{\rho} d\rho \right]$ *and*
$M = \|u_+\|_{L^2(Q_1^-)} + \|u_-\|_{L^2(Q_1^-)}$.

A Variant of the Almost Monotonicity Formula

Theorem (Matevosyan-Petrosyan)

Let $u_\pm$ *satisfy* $u_\pm \geq 0$, $\mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1$, $u_+ \cdot u_- = 0$ *in* S_1, *and*

$$u_\pm(x,s) \leq \sigma((|x|^2 + |s|)^{1/2}) \quad \textit{for } (x,s) \in Q_1^-$$

for a Dini modulus of continuity $\sigma(r)$. *Then* $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ *satisfies*

$$\Phi(r) \leq [1 + \alpha(\rho)]\Phi(\rho) + C_{M,\psi,\sigma,\omega}\alpha(\rho), \quad 0 < r \leq \rho \leq r_\omega,$$

where $\alpha(r) = C_0 \left[r + \sigma(r^{1/2}) + \int_0^r \frac{\sigma(\rho^{1/2})}{\rho} d\rho + \int_0^r \frac{\theta(\rho)}{\rho} d\rho \right]$ *and*
$M = \|u_+\|_{L^2(Q_1^-)} + \|u_-\|_{L^2(Q_1^-)}$.

- This guaranties the existence of $\Phi(0+) = \lim_{r \to 0+} \Phi(r)$.

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x, u, \nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x,u,\nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.
- For $x_0 \in \partial\Omega$ (free boundary) consider *rescalings*

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0 + rx) - u(x_0)}{r^2}.$$

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x,u,\nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.
- For $x_0 \in \partial\Omega$ (free boundary) consider *rescalings*

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0+rx) - u(x_0)}{r^2}.$$

- Limits of u_r over $r = r_j \to 0+$ are called *blowups* of u at x_0

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x, u, \nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.
- For $x_0 \in \partial\Omega$ (free boundary) consider *rescalings*

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0 + rx) - u(x_0)}{r^2}.$$

- Limits of u_r over $r = r_j \to 0+$ are called *blowups* of u at x_0
- Key question: what are the possible blowups?

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x, u, \nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.
- For $x_0 \in \partial\Omega$ (free boundary) consider *rescalings*

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0 + rx) - u(x_0)}{r^2}.$$

- Limits of u_r over $r = r_j \to 0+$ are called *blowups* of u at x_0
- Key question: what are the possible blowups?

Theorem (Matevosyan-Petrosyan)

The blowups are either **one-dimensional** *or* **quadratic polynomial**.

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x,u,\nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.
- For $x_0 \in \partial\Omega$ (free boundary) consider *rescalings*

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0 + rx) - u(x_0)}{r^2}.$$

- Limits of u_r over $r = r_j \to 0+$ are called *blowups* of u at x_0
- Key question: what are the possible blowups?

Theorem (Matevosyan-Petrosyan)

The blowups are either **one-dimensional** *or* **quadratic polynomial**.

- One dimensional means $u_0(x) = v(x \cdot e_0)$ for some direction e_0

Application: Classification of Blowups

- Let u solve $\operatorname{div}(a(|\nabla u|^2)\nabla u) = f(x, u, \nabla u)\chi_\Omega$, $|\nabla u| = 0$ on Ω^c.
- For $x_0 \in \partial\Omega$ (free boundary) consider *rescalings*

$$u_r(x) = u_{x_0,r}(x) = \frac{u(x_0 + rx) - u(x_0)}{r^2}.$$

- Limits of u_r over $r = r_j \to 0+$ are called *blowups* of u at x_0
- Key question: what are the possible blowups?

Theorem (Matevosyan-Petrosyan)

The blowups are either **one-dimensional** *or* **quadratic polynomial**.

- One dimensional means $u_0(x) = v(x \cdot e_0)$ for some direction e_0
- Equivalently, $\partial_e u_0$ has a sign in $\mathbb{R}^n$ for any direction e.

Application: Classification of Blowups

Idea of the proof (assuming $x_0 = 0$)

- Recall that $\mathcal{L}_{\mathcal{A},b,c}(\partial_e u)^{\pm} \geq -M$ for any direction e

Application: Classification of Blowups

Idea of the proof (assuming $x_0 = 0$)

- Recall that $\mathcal{L}_{\mathcal{A},b,c}(\partial_e u)^\pm \geq -M$ for any direction e
- We also have that $|(\partial_e u)^\pm(x)| \leq C|x|^\alpha$

Application: Classification of Blowups

Idea of the proof (assuming $x_0 = 0$)

- Recall that $\mathcal{L}_{\mathcal{A},b,c}(\partial_e u)^{\pm} \geq -M$ for any direction e
- We also have that $|(\partial_e u)^{\pm}(x)| \leq C|x|^{\alpha}$
- Thus, $\varphi(0+, (\partial_e u)^{+}, (\partial_e u)^{-}) = c_0$ exists.

Application: Classification of Blowups

Idea of the proof (assuming $x_0 = 0$)

- Recall that $\mathcal{L}_{\mathcal{A},b,c}(\partial_e u)^\pm \geq -M$ for any direction e
- We also have that $|(\partial_e u)^\pm(x)| \leq C|x|^\alpha$
- Thus, $\varphi(0+, (\partial_e u)^+, (\partial_e u)^-) = c_0$ exists.
- If $u_{r_j} \to u_0$ in $W^{2,p}$, then we have

$$\begin{aligned}\varphi(r, (\partial_e u_0)^+, (\partial_e u_0)^-) &= \lim_{j\to\infty} \varphi(r, (\partial_e u_{r_j})^+, (\partial_e u_{r_j})^-)\\ &= \lim_{j\to\infty} \varphi(r r_j, (\partial_e u)^+, (\partial_e u)^-)\\ &= c_0\end{aligned}$$

i.e. $\varphi(r, (\partial_e u_0)^+, (\partial_e u_0)^-) \equiv const$

Application: Classification of Blowups

Idea of the proof (assuming $x_0 = 0$)

- Recall that $\mathcal{L}_{\mathcal{A},b,c}(\partial_e u)^{\pm} \geq -M$ for any direction e
- We also have that $|(\partial_e u)^{\pm}(x)| \leq C|x|^{\alpha}$
- Thus, $\varphi(0+, (\partial_e u)^+, (\partial_e u)^-) = c_0$ exists.
- If $u_{r_j} \to u_0$ in $W^{2,p}$, then we have

$$\begin{aligned}\varphi(r, (\partial_e u_0)^+, (\partial_e u_0)^-) &= \lim_{j\to\infty} \varphi(r, (\partial_e u_{r_j})^+, (\partial_e u_{r_j})^-)\\ &= \lim_{j\to\infty} \varphi(r r_j, (\partial_e u)^+, (\partial_e u)^-)\\ &= c_0\end{aligned}$$

  i.e. $\varphi(r, (\partial_e u_0)^+, (\partial_e u_0)^-) \equiv const$
- Problem is reduced to analyzing the case of equality for the original Alt-Caffarelli-Friedman montonicity formula (Caffarelli-Karp-Shahgholian 2000)