Some results on BSDEs and related HJB equations

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PLAN

1. Statement of the problem.

2. HJB equations with superquadratic hamiltonian: lipschitz continuous final datum.

3. HJB equations with quadratic hamiltonian: continuous final datum.

4. Application to stochastic optimal control.

PDE in an infinite dimensional Hilbert space H

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{L}v\,(t,x) + \psi\left(\nabla v\,(t,x)\sqrt{Q}\right) + l(x), & t \in [0,T]\,,\ x \in H \\ v(T,x) = \phi\,(x)\,, \end{cases}$$

$$(\mathcal{L}f)(x) = \frac{1}{2}(TrQ\nabla^2 f)(x) + \langle Ax, \nabla f(x)\rangle.$$

$\mathcal{L}$ generator of the transition semigroup P_t of the Ornstein-Uhlenbeck process

$$\begin{cases} dX_t = AX_t dt + \sqrt{Q}dW_t, & t \in [0,T] \\ X_0 = x. \end{cases}$$

F. Gozzi, Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities, JMAA 1996.

X. Bao, F. Delbaen and Y. Hu, Backward SDEs with superquadratic growth, arXiv:0902.3316, to appear on PTRF.

Assumptions on A, Q, P_t.

1. A generator of a C_0 semigroup $\left(e^{tA}, t \geq 0\right)$ in H. $\exists M \geq 1$, $\omega \geq 0$ s.t. $\|e^{tA}\|_{L(H,H)} \leq M e^{\omega t}$.

2. $\sqrt{Q} : H \to H$ and $Q_\sigma := \int_0^\sigma e^{sA} Q^* e^{sA^*} ds$ of trace class $\forall\, \sigma \geq 0$.

3. regularizing property for P_t: for some $\alpha \in [0,1)$ and $\forall\, \phi \in C_b(H)$, $\exists c > 0$ s.t. $\forall \xi \in H$ and for $0 \leq t < \tau \leq T$,

$$\left| \nabla P_\tau [\phi](x) \sqrt{Q} \xi \right| \leq \frac{c}{\tau^\alpha} \|\phi\|_\infty |\xi|$$

4. A and Q commute: point 3 holds true with $\alpha = 1/2$.

see F. M., Semilinear Kolmogorov equations and applications to stochastic optimal control, AMO 2005.

Linear heat equation

$\mathcal{O} \in \mathbb{R}$, $H = L^2(\Omega)$, $\{e_k\}_{k\in\mathbb{N}}$ complete orthonormal basis diagonalizing Δ, endowed with Dirichlet boundary conditions in $\mathcal{O}$.

$$\begin{cases} \dfrac{\partial y}{\partial s}(s,\xi) = \Delta y(s,\xi) + \dfrac{\partial W^Q}{\partial s}(s,\xi), \qquad s \in [t,T],\ \xi \in \mathcal{O}, \\ y(t,\xi) = x(\xi), \\ y(s,\xi) = 0, \quad \xi \in \partial\mathcal{O}. \end{cases}$$

$W^Q(s,\xi)$ Gaussian mean zero random field, with covariance given by

$$\mathbb{E} < W^Q(s,\cdot), h >_H < W^Q(t,\cdot), k >_H = t \wedge s < Qh, k >_H .$$

• $Q : H \to H$ positive, $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$, formally $W^Q(s,\cdot) = \sum_{k\geq 0} Qe_k(\cdot)\beta_k(s)$. with $\{\beta_k(s)\}_{k\in\mathbb{N}}$ independent Brownian motions.

"Abstract" formulation of the heat equation

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q} dW_\tau, \qquad \tau \in [t,T] \\ X_t = x. \end{cases}$$

HJB equations with superquadratic hamiltonian

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{L}v\,(t,x) + \psi\left(\nabla v\,(t,x)\,\sqrt{Q}\right) + l(x), & t \in [0,T]\,,\ x \in H \\ v(T,x) = \phi\,(x)\,. \end{cases}$$

Hypothesis $\psi : H \to \mathbb{R}$, is Gâteaux differentiable and for every $\xi_1, \xi_2 \in H$, $|\psi(\xi_1) - \psi(\xi_2)| \leq (1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2|$, for $p \geq 2$.

Mild solution $v \in C_b\,([0,T] \times H, \mathbb{R})$ s.t. $|\nabla v(t,x)\sqrt{Q}| \leq Ct^{-\alpha}$ satisfying

$$v(t,x) = P_{t,T}\,[\phi]\,(x) - \int_t^T P_{t,s}\left[\psi(\nabla v\,(s,\cdot)\,\sqrt{Q})\right](x)\,ds, - \int_t^T P_{t,s}\,[l]\,(x)\,ds.$$

From now on we present results in the case $l = 0$ and $\psi(0) = 0$.

Related forward-backward SDE (FBSDE)

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q} dW_\tau, \quad \tau \in [t,T] \subset [0,T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau)\, d\tau + Z_\tau\, dW_\tau, \\ Y_T = \phi(X_T). \end{cases}$$

Let us also denote $(X,Y,Z) = (X^{t,x}, Y^{t,x}, Z^{t,x})$.

Connection between BSDEs and HJBs

If v solve HJB equation, then $(X_\tau^{t,x}, v(\tau, X_\tau^{t,x}), \nabla v(\tau, X_\tau^{t,x}))$ solve FBSDE.

Viceversa if $(X^{t,x}, Y^{t,x}, Z^{t,x})$ solve the FBSDE then $v(t,x) := Y_t^{t,x}$ is mild solution to HJB, provided $Z_t^{t,x} = \nabla_x Y_t^{t,x} \sqrt{Q}$.

E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems and Control Lett. 1990.
M. Fuhrman, G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. AoP 2002.

Theorem 1 *If the previous assumptions on A, Q and ψ hold true, and if ϕ is bounded and lipschitz continuous, then equation*

$$\begin{cases} dY_\tau = -\psi(Z_\tau)\, d\tau + Z_\tau\, dW_\tau, \\ Y_T = \phi(X_T), \end{cases}$$

admits a unique solution. The function $v(t,x) = Y_t^{t,x}$ is the unique mild solution of the HJB equation

Sketch of the proof. At first we assume further that ϕ is Gâteaux differentiable with bounded derivative.

• Local mild solution on $[T-\delta, T]$ s.t., due to the regularity of ϕ, $\nabla v(t,x)\sqrt{Q}$ does not blow up: namely

$$|v(t,x)| + \|\nabla v(t,x)\sqrt{Q}\| \leq 2Me^{\omega T}\|\phi\|_1$$

• Equivalent representation of the mild solution:

$$v(t,x) = R_{t,T}\left[\phi\right](x)\,, \quad t \in \left[T-\delta, T\right],\ x \in H.$$

Set $G(t,x) = \int_0^1 \nabla\psi(\lambda\nabla^{\sqrt{Q}}v(t,x))d\lambda$.

$R_{t,\tau}$ transition semigroup of $\Theta_{\cdot}^{t,x}$, solution to

$$\begin{cases} d\Theta_\tau^{t,x} = A\Theta_\tau^{t,x}d\tau + \sqrt{Q}G(\tau,\Theta_\tau^{t,x})d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \qquad t \in [T-\delta,T] \\ \Theta_t = x, & \end{cases}$$

• A priori estimates: by the equivalent representation v is bounded. By the connection with BSDEs we also get $\nabla v\sqrt{Q}$ is bounded.

• Global existence follows by local existence and apriori estimates.

• ϕ bounded and lipschitz continuous: let the inf-sup convolution of ϕ

$$\phi_n(x) = \sup_{z\in H}\left\{\inf_{y\in H}\left[\phi(y) + n\frac{|z-y|_H^2}{2}\right] - n\,|x-z|_H^2\right\}$$

L lipschitz constant of $\phi \Rightarrow |\nabla\phi_n| \le L$.

Aim: solve the HJB equation with quadratic hamiltonian ψ and with final datum ϕ bounded and continuous. Fundamental assumption: A and Q commute.

• In the case of Gâteaux differentiable final datum ϕ find an estimate for the $\nabla v(t,x)\sqrt{Q}$ depending on T, t, $\|\phi\|_\infty$ but not on $\nabla\phi$:

$$|\nabla v(t,x)\sqrt{Q}\xi| \leq C(T-t)^{-1/2}.$$

BMO martingales technique abd a new Bismut-Elworthy formula, see

X. Bao, F. Delbaen and Y. Hu, and

A. Richou, Numerical simulation of BSDEs with drivers of quadratic growth, arXiv:1001.0401.

• ϕ bounded and continuous: approximation procedure on ϕ by means of its inf-sup convolutions $(\phi_n)_n$.

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = -\mathcal{L}v\,(t,x) + \psi\left(\nabla v\,(t,x)\sqrt{Q}\right) + l(x), & t \in [0,T]\,,\ x \in H \\ v(T,x) = \phi\,(x)\,. \end{cases}$$

$$\begin{cases} dX_\tau = AX_\tau d\tau + \sqrt{Q}dW_\tau, \quad \tau \in [t,T] \subset [0,T], \\ X_t = x, \\ dY_\tau = -\psi(Z_\tau)\,d\tau + Z_\tau\,dW_\tau, \\ Y_T = \phi(X_T^{t,x}). \end{cases}$$

Hypothesis $\psi : H \to \mathbb{R}$, is Gâteaux differentiable and for every $\xi_1, \xi_2 \in \Xi$, $|\psi(\xi_1) - \psi(\xi_2)| \leq (1 + |\xi_1| + |\xi_2|)|\xi_1 - \xi_2|$.

Theorem 2 *Let (Y,Z) be the solution of the BSDE. If A, Q, ψ satisfy the previous assumptions, and if ϕ is differentiable then the following estimate holds true:*

$$|Z_t^{t,x}| \leq C\|\phi\|_\infty (T-t)^{-1/2},$$

where C depends on t, T, A and not on $\nabla\phi$. Moreover $Z_t^{t,x} = \nabla v(t,x)\sqrt{Q}$.

Sketch of the proof. $(\Phi(\tau)=\int_t^\tau Z_s dW_s)_{\tau\in[t,T]}$ BMO martingale

$$\|\Phi\|_{BMO}=\sup_{\sigma\in[t,T]}\mathbb{E}\left[\int_\sigma^T Z_s^2 ds|\mathcal{F}_\sigma\right]^{1/2}<+\infty,$$

$F_\tau^{t,x}=\nabla Y_\tau^{t,x}\sqrt{Q}h$ and $V_\tau^{t,x}=\nabla Z_\tau^{t,x}\sqrt{Q}h$.

$$\begin{cases} dF_\tau^{t,x}=-\nabla\psi(Z_\tau^{t,x})V_\tau^{t,x}\,d\tau+V_\tau^{t,x}\,dW_\tau, \\ F_T^{t,x}=\nabla\phi(X_T^{t,x})e^{(T-t)A}\sqrt{Q}h, \end{cases}$$

$\mathbb{Q}$ s.t. $W_\tau^{\mathbb{Q}}:=-\int_0^\tau\nabla\psi(Z_s^{t,x})ds+W_\tau$ is a Wiener process.

$\left(\int_t^\tau\nabla\psi(Z_s^{t,x})dW_s\right)_{\tau\in[t,T]}$ BMO martingale

$$\mathcal{E}_\tau=\exp\left(\int_t^\tau\psi(Z_s^{t,x})W_s-\frac{1}{2}\int_t^\tau|\psi(Z_s^{t,x})|^2ds\right),$$

is a uniformly integrable martingale.

In $(\Omega,\mathcal{F},\mathbb{Q})$, $F^{t,x}$ is a martingale, so $(F^{t,x})^2$ is a $\mathbb{Q}$-submartingale.

So

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |F_s^{t,x}|^2 ds|\mathcal{F}_t\right] \geq |F_t^{t,x}|^2(T-t) = |Z_t^{t,x}h|^2(T-t).$$

A and Q commute,

$$F_\tau^{t,x} :=< \nabla v(\tau, X_\tau^{t,x}), \sqrt{Q}h >=< \nabla_x v(\tau, X_\tau^{t,x}), e^{(\tau-t)A}\sqrt{Q}h >$$
$$=< \nabla_x v(\tau, X_\tau^{t,x}), \sqrt{Q}e^{(\tau-t)A}h >=< Z_\tau^{t,x}, e^{(\tau-t)A}h > .$$

Moreover

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T |F_s^{t,x}|^2 ds|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T | < Z_s^{t,x}, e^{(s-t)A}h > |^2 ds|\mathcal{F}_t\right] \leq C_{t,T},$$

where $C_{t,T}$ depends only on T, t, A. So

$$|Z_t^{t,x}| \leq C(T-t)^{-1/2},$$

where C depends on t, T, A, $\|\phi\|_\infty$ and not on $\nabla\phi$.

Approximation procedure

Set v_n solution of

$$\begin{cases} \frac{\partial v_n}{\partial t}(t,x) = -\mathcal{L}v_n\,(t,x) + \psi\left(\nabla v_n\,(t,x)\,\sqrt{Q}\right), & t \in [0,T]\,,\ x \in H \\ v_n(T,x) = \phi_n\,(x)\,, & \end{cases}$$

ϕ_n inf-sup convolution of ϕ.

$v_n(t,x) = Y_t^{n,t,x}$, $\nabla v_n(t,x)\sqrt{Q} = Z^{n,t,x}$, where (Y^n, Z^n) solve.

$$\begin{cases} dY_\tau^{n,t,x} = -\psi(Z_\tau^{n,t,x})\, d\tau + Z_\tau^{n,t,x}\, dW_\tau, \\ Y_T^{n,t,x} = \phi_n(X_T^{t,x}). \end{cases}$$

- As $n \to +\infty$, $(Y^n, Z^n) \to (Y, Z)$.
- Show that $Z_t^{t,x} = \nabla v(t,x)\sqrt{Q}$.

Theorem 3 *If A, Q, ψ satisfy the previous assumptions, and if ϕ is bounded and continuous, then the HJB equation admits a unique mild solution.*

controlled state equation

$$\begin{cases} dX_\tau^u = \left[AX_\tau^u + \sqrt{Q}u_\tau\right]d\tau + \sqrt{Q}dW_\tau, & \tau \in [t,T] \\ X_t^u = x. \end{cases}$$

cost and value function

$$J(t,x,u) = \mathbb{E}\int_t^T [l(X_s^u) + g(u_s)]ds + \mathbb{E}\phi(X_T^u), \qquad J^*(t,x) = \inf_u J(t,x,u).$$

where for some $1 < q \le 2$ $\exists$ $R, c > 0$ s. t.

$$0 \le g(u) \le c(1 + |u|^q), \quad g(u) \ge c|u|^q \qquad \text{for every } u : |u| \ge R.$$

Admissible controls: predictable processes s. t. $\mathbb{E}\int_0^T |u_t|^q dt < +\infty$.
Hamiltonian function $\psi(z) = \inf_u \{g(u) + zu\} \quad \forall z \in H$.

v solution of the corresponding HJB $\Rightarrow$ $J^*(t,x) = v(t,x)$, optimal control is given in feedback form.