Semilinear perturbations of Kolmogorov operators, obstacle problems, and optimal stopping

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Based on joint work with Viorel Barbu and Zeev Sobol

Outline

1. Motivation and history of the problem

- Classical: the price of an American option is (often) the solution of an obstacle problem
- Not so classical (Kholodnyi NA '97): the price of an American option is expected to "solve" a semilinear PDE with a discontinuous reaction term
- Q: Is there a general connection among optimal stopping, obstacle problems, variational inequalities, semilinear PDEs?

2. Solving obstacle problems via semilinear PDEs

- Solution of a semilinear PDE is the solution of the obstacle problem
- (Nonlinear) monotone operator techniques give a natural concept of solution
- Equation is globally well-posed

3. Back to optimal stopping

- Solution of the semilinear PDE is also (often) the value function of the original optimal stopping problem

Starting point: 1D Black-Scholes heuristics

Assume Black-Scholes dynamics

$$dS_t = (r-d)S_t\,dt + \sigma S_t dW_t$$

and an American option with payoff $g : \mathbb{R}_+ \to \mathbb{R}_+$, with price

$$v(t,x) = \sup_{\tau \in [t,T]} \mathbb{E}_{t,x} e^{-r(\tau - t)} g(S_\tau)$$

(Bensoussan AAM '84, Karatzas AMO '88)

By heuristic (clever!) arguments, Kholodnyi "showed" that v solves the PDE

$$v_t + \frac{1}{2}\sigma^2 x^2 v_{xx} + (r-d)xv_x - rv = q(x,v), \qquad v(T,x) = g(x),$$

where

$$q(x,v) = \begin{cases} -d(x), & v \le g(x), \\ 0, & v > g(x), \end{cases}$$

$$d(x) = \Big(-\frac{1}{2}\sigma^2 x^2 g_{xx} - (r-d)xg_x + rg \Big)^+.$$

Issues with Kholodnyi's equation

- ★ What do we mean by "solution"? The reaction term q is discontinuous (if at all defined)
- ★ The space domain is unbounded, coefficients are unbounded
- ★ d is in general defined only as a distribution in $\mathcal{D}'(\mathbb{R})$

As far as we know, no known technique applies in the general case:

- ✘ PDE with discontinuous terms: does not handle unbounded coefficients
- ✘ PDE with growing coefficients: does not handle discontinuous terms
- ✘ PDE with measure data: does not handle "rough" equations
- ✔ Viscosity solutions approach for call/put options on a single BS asset works! (Benth et al. F&S '03)

Semilinear PDEs vs. Variational Inequalities

Classical analytic approach: solve the obstacle problem

$$\begin{cases} v_t + L_0 v - cv \leq f, \quad v_t + L_0 v - cv = f \quad \text{on } \{v(t,x) > g(x)\} \\ v(T,x) = g(x) \end{cases}$$

e.g. turning it into a VI

$$\frac{dv}{dt} + L_0 v - cv - \mathcal{N}_g(v) \ni f, \quad v(T) = g(T).$$

Why another approach?

- Much easier to do numerical analysis on a PDE rather than on a free boundary problem/VI (Benth et al. IFB '04)
- Nonlinear discontinuous PDEs have an intrinsic mathematical interest
- New way to solve obstacle problems without a variational setting
- Can one do better than call/put options on one BS asset?

An abstract general framework

Let X be a right Markov process on a Hilbert space H, with semigroup P_t, and consider the optimal stopping problem

$$v(t,x) = \sup_{\tau \in [t,T]} \mathbb{E}_{t,x} e^{-\int_t^\tau c(X_s)\,ds} g(X_\tau)$$

Goal: characterization of the value function v in terms of the solution of a suitable semilinear equation.

Plan:

- Construct a state space E
- Formulate abstract semilinear (evolution) eq. on E
- Specify the concept of solution
- Prove well-posedness
- Prove that the solution coincides with the solution of the obstacle problem
- Prove that the solution coincides with the value function

State space: $L^p(H,\nu)$, $p \geq 1$

Let ν be an excessive probability measure for P_t with full topological support, i.e. such that

$$\int_H P_t\varphi\, d\nu \leq e^{\omega t} \int_H \varphi\, d\nu \qquad \forall \varphi \in C_b(H)^+,$$

and $\nu(U) > 0$ for any nonempty open set $U \subseteq H$.

Such a measure ν always exists! (Röckner-Trutnau IDAQP '07)

Set $A = -N_p + cI$, where $-N_p$ is the generator of P_t on $L^p(H,\nu)$. Then $-N_p$ (hence also A) is ω-m-accretive, i.e. (let $\omega = 0$ for simplicity)

(i) $\langle x, J(y)\rangle \geq 0 \quad \forall [x,y] \in A$;

(ii) $R(\lambda I + A) = L^p(E,\nu) \quad \forall \lambda > 0$.

The semilinear PDE

Given $g \in L^p(H,\nu)$ and $d \in L^p(H,\nu)^+$ define the nonlinear multivalued operator B_d on $L^p(H,\nu)$ by

$$[B_d y](x) = \begin{cases} -d(x), & y(x) < g(x), \\ [-d(x), 0], & y(x) = g(x), \\ 0, & y(x) > g(x). \end{cases}$$

We are going to establish well-posedness in $L^p(E,\nu)$ of the evolution equation

$$\frac{du}{dt}(t) + Au(t) + B_d u(t) \ni 0, \qquad u(0) = g. \tag{1}$$

We are to "use" several concepts of solutions:

- Strong: (1) satisfied a.e. on $(0, T)$
- Generalized: SOLA
- Mild in the sense of Crandall-Liggett: limit of a discrete scheme
- Mild in the sense of Duhamel's principle

Well-posedness

Theorem. The equation

$$\frac{du}{dt}(t) + Au(t) + B_d u(t) \ni 0, \qquad u(0) = g$$

admits a unique CL-mild solution in $L^p(E, \nu)$, $p \geq 1$ (which is SOLA for $p > 1$).
Moreover, if $g \in D(A)$ and $p > 1$, then it admits a unique strong solution

$$u \in W^{1,\infty}([0, T] \to L^p(H, \nu)) \cap L^\infty([0, T] \to D(A))$$

which is also right-differentiable.

Proof. Have to show that $A + B_d$ is ω-m-accretive...

$A + B_d$ is ω-m-accretive in $L^p(H, \nu)$

Lemma. B_d is m-accretive in $L^p(H, \nu)$.

Proof. B_d is accretive because

$$y \mapsto d(x)(\mathfrak{H}(y - g(x)) - 1)$$

is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ for each $x \in H$. Maximality: equation $y + By = f$, $f \in L^p(H, \nu)$ easily admits a solution.

Theorem. $A + B_d$ is ω-m-accretive in $L^p(H, \nu)$.

Proof. Three different cases:

- $p = 2$: follows by Rockafellar's criterion: $D(A) \cap \operatorname{int} D(B_d) \neq \varnothing$.
- $p > 1$: solve $u_\lambda + A_\lambda u_\lambda + B_d u_\lambda \ni f$, get a priori estimates on u_λ, let $\lambda \to 0$ (reflexivity of L^p is crucial)
- $p = 1$: solve $u_\varepsilon + Au_\varepsilon + B_{d,\varepsilon} u_\varepsilon = f$, get monotonicity for u_ε, let $\varepsilon \to 0$.

Solution of the PDE is a solution of the obstacle problem

Have to choose d first! Two cases:

Theorem.

- If $g \in D(A)$, let $d := (Ag)^+$.
- If $g \in L^p(H,\nu) = \overline{D(A)}$, assume that $(A_\lambda g)^+$ is weakly compact in $L^p(H,\nu)$, and let d be such that $(A_\lambda g)^+ \rightharpoonup d$.

Then $u(t) \geq g$ ν-a.s..

Proof. Prove that $S(t) := e^{-t(A+B_d)}$ leaves invariant

$$\mathcal{K}_g := \left\{\varphi \in L^p(H,\nu) : \ \varphi \geq g \quad \nu\text{-a.e.}\right\}.$$

Enough to prove that $(I + \lambda A + \lambda B_d)^{-1}\mathcal{K}_g \subseteq \mathcal{K}_g$ for all $\lambda \in]0, \omega^{-1}[$. Use sub-Markovianity of A and definition of B_d.

Key observation: By definition of B_d, $u(t) \geq g$ ν-a.e. implies that u is *the* solution of the obstacle problem!

Solution as value function of the optimal stopping problem

Theorem 1. Assume that P_t is strong Feller (or $\mathcal{L}(X_t) \ll \nu$). Then

$$u(T-t) = v(t) \qquad \nu\text{-a.e.} \quad \forall t \leq T$$

Proof. Two steps:

1. Establish a Duhamel representation for CL-mild solutions.
2. Refine the proof in Barbu-M (AMO '08)

Remark. Still true for Markov processes that are limits of strong Feller processes (hence always true for solutions of SDEs on $\mathbb{R}^d$)

Q:

- Can one approximate any right Markov process by a strong Feller Markov process?
- Counterexamples?

Further properties

- Under very mild assumptions v is continuous, hence u has a continuous ν-modification. Moreover, recall that ν has full topological support!
- Convexity assumptions on g are needed also for the 1D viscosity solutions approach.
- Finite-difference schemes converge to the mild solution (Crandall-Liggett theorem)
- If $\dim H < \infty$ one expects further regularity for u
- Infinite-horizon optimal stopping problems ($\rightsquigarrow$ elliptic PDEs) are automatically included

An example

Let X be the solution of an SDE on H

$$X_s = x + \int_t^s b(X_r)\,dr + \int_t^s \sigma(X_r)\,dW(r)$$

with Kolmogorov operator

$$-N_0\phi = \frac{1}{2}\,\mathrm{Tr}[(\sigma Q^{1/2})(\sigma Q^{1/2})^* D^2\phi] + \langle b(x), D\phi\rangle_H\,, \qquad \phi \in C_b^2(H).$$

Consider an optimal stopping problem as before.

Let $P_t\phi(x) := \mathbb{E}_x\phi(X(t))$, $\phi \in C_b(H)$. Let $P_t^*\nu \leq e^{\omega t}\nu$, extend P_t to $L^2(H,\nu)$, and set $-N_2\phi := \lim_{h\downarrow 0} h^{-1}(P_h\phi - \phi)$ in $L^2(H,\nu)$.

Lemma. Let $b \in C^2(H) \cap L^2(H,\nu)$, $\sigma \in C^2(H, L(H,H))$, and

$$|Db(x)|_H + |D\sigma(x)|_{L(H,H)} \leq C \qquad \forall x \in H.$$

Then $-N_0$ is ω-accretive and $-N_2$ is the closure in $L^2(H,\nu)$ of $-N_0$ defined on $D(N_0) = C_b^2(H)$.

(contd.)

Proof. $-N_0$ is ω-accretive, hence closable. Fix $f \in C_b^2(H)$ and consider the eq. $(\lambda I + N_0)\varphi = f$. Candidate solution is

$$\varphi(x) = \mathbb{E} \int_0^\infty e^{-\lambda t} f(X_t^x)\, dt, \qquad \lambda > \omega.$$

By second order differentiability of $x \mapsto X_t^x$ and Itô's formula, we see that ϕ actually is the solution. Then $R(I + \lambda N_0) \subset L^2(H, \nu)$ densely, so $\overline{-N_0}$ is ω-m-accretive by Lumer-Phillips theorem, so it must be $N_2 = \overline{N_0}$ because $-N_2$ is also ω-m-accretive.

Remarks.

- Similar results for Kolmogorov operators go through under much weaker assumptions, and also for equations of the type

$$X_s = x + \int_t^s b(X_r)\, dr + \int_t^s \sigma(X_r)\, dW(r) + \int_t^s \int_Z g(z, X_{r-})\, \bar{\mu}(dz, dr)$$

  (M-Prévôt-Röckner JFA '10)
- Enough to take $N_0^* \nu \le \omega \nu$.

American options with general volatility I

Let $\mathbb{Q}$ be EMM, n assets with price-per-share X_t,

$$dX_t = rX_t\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x \geq 0, \quad t \in [0, T],$$

W is $\mathbb{R}^m$-valued Wiener process, and $\sigma : \mathbb{R}^n \to L(\mathbb{R}^m, \mathbb{R}^n)$ is the volatility function.

We do *not* assume that σ satisfies any nondegeneracy condition!

Pricing an American contingent claim with payoff $g : \mathbb{R}^n \to \mathbb{R}$ is equivalent to the optimal stopping problem

$$v(t,x) = \sup_{\tau \in [t,T]} \mathbb{E}_{t,x}[e^{-r\tau} g(X_\tau)],$$

Kolmogorov operator:

$$-N_0 f(x) = \frac{1}{2}\,\mathrm{Tr}[\sigma(x)\sigma^*(x)D^2 f(x)] + \langle rx, Df(x)\rangle_{\mathbb{R}^n}, \qquad f \in C_b^2(\mathbb{R}^n).$$

Classical (analytic) approach (VI in Sobolev spaces w.r.t. Lebesgue measure) does *not* apply, essentially because σ can be degenerate.

American options with general volatility II

Lemma. Assume that

$$\sigma \in C^2(\mathbb{R}^n), \quad |\sigma_{x_i}| + |\sigma_{x_i x_j}| \le C.$$

Then there exists an excessive probability measure ν for P_t of the form

$$\nu(dx) = \frac{a}{1 + |x|^{2(n+1)}}\, dx.$$

Proof. Try to solve $N_0^* \rho \le \omega \rho$ in $\mathcal{D}'(\mathbb{R}^n)$ ...

Lemma. The infinitesimal generator of P_t in $L^2(\mathbb{R}^n, \nu)$ is $-N_2 := \overline{-N_0}$. Moreover one has

$$\int_{\mathbb{R}^n} (N_0 f) f \, d\nu \le -\frac{1}{2} \int_{\mathbb{R}^n} |\sigma^* Df|^2 \, d\nu + \omega \int_{\mathbb{R}^n} f^2 \, d\nu \qquad \forall f \in C_b^2(\mathbb{R}^n).$$

American options with general volatility III

We can now apply the abstract existence results: v is uniquely identified by the mild solution in $L^2(\mathbb{R}^n, \nu)$ of the nonlinear parabolic equation

$$\frac{du}{dt} + N_2 u + ru + Bu \ni 0, \quad u(0) = g,$$

provided g satisfies

$$Dg \in L^\infty(\mathbb{R}^n), \quad \mathrm{Tr}[\sigma\sigma^* D^2 g] \in \mathcal{M}(\mathbb{R}^n), \quad \mathrm{Tr}[\sigma\sigma^* D^2 g] \geq 0 \text{ in } \mathcal{M}(\mathbb{R}^n).$$

- Exchange options and basket put options can be covered as examples.
- American options on assets with stochastic volatility can be treated in a similar way (augmenting the state space).
- Asian options with American feature also covered (adding an auxiliary process)

$$v(x) = \sup_{\tau \in [0,T]} \mathbb{E}_x \left(K - \frac{1}{\tau + \delta} \int_0^\tau X_s \, ds \right)^+ .$$

Path-dependent American options I

Price of n assets under an EMM $\mathbb{Q}$:

$$\begin{cases} dX(t) = rX(t) + \sigma(X(t), X_s(t))\, dW(t), & 0 \le t \le T \\ X(0) = x_0, \qquad X_s(0) = x_1(s), & -T \le s \le 0, \end{cases}$$

where $X_s(t) := X(t+s)$, $s \in (-T, 0)$.

Consider an American option with payoff $g : \mathbb{R}^n \times L^2([-T, 0]) \to \mathbb{R}$:

$$v(s, x_0, x_1) = \sup_{\tau \in [0,T]} \mathbb{E}_{s,(x_0,x_1)}[e^{-r\tau} g(X(\tau), X_s(\tau))],$$

e.g. $g(x_0, x_1) = \alpha_0 g_0(x_0) + \alpha_2 g_1(x_1)$, with α_1, $\alpha_2 \ge 0$ and

$$g_0(x_0) = (K_0 - x_0)^+, \qquad g_1(x_1) = \left(K_1 - \int_{-T}^{0} x_1(s)\, ds\right)^+$$

Path-dependent American options II

- Formulate as an infinite dimensional problem in $H = \mathbb{R}^n \times L^2([-T, 0], \mathbb{R}^n)$
- Look for an excessive measure ν of the form $\nu = \nu_1 \otimes \nu_2$, with ν_1, ν_2 probability measures on $\mathbb{R}^n$ and $L^2([-T, 0], \mathbb{R}^n)$.
  Choose $\nu_1(dx_0) = \rho(x_0)\, dx_0$,

  $$\rho(x_0) = \frac{a}{1 + |x_0|^{2n}}, \qquad a = \left(\int_{\mathbb{R}^n} \frac{1}{1 + |x_0|^{2n}}\, dx_0 \right)^{-1},$$

  and ν_2 a Gaussian measure on $L^2([-T, 0], \mathbb{R}^n)$.
- Impose assumptions on σ so that $\overline{N_0} = N_2$ in $L^2(H, \nu)$ (but still no nondegeneracy!)
- Impose assumptions on g such that $(A_\lambda g)^+$ w.c. in $L^2(H, \nu)$.

More questions

- Interest rates derivatives can be covered (done for Musiela's dynamics only) – what about BGM?
- Excessive measures for general S(P)DEs with jumps are difficult to "guess"
- General results on existence of smooth excessive measure?
- A nonlinear?
- Viscosity solutions in higher dimensions?
- Is the viscosity solution a mild (or generalized) solution? Viceversa?