Dirichlet problems for Ornstein-Uhlenbeck operators in subsets of Hilbert spaces

Alessandra Lunardi
in collaboration with G. Da Prato

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$$\mathcal{L}_\alpha\varphi(x) = \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^{1-\alpha}D_{kk}\varphi(x) - \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^{-\alpha}x_k D_k\varphi(x).$$

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The associated transition Markov semigroups $T_\alpha(t)$ are the Ornstein-Uhlenbeck semigroups

$$T_\alpha(t)\varphi(x) = \mathbb{E}[\varphi(X_\alpha(t,x))] = \int_H \varphi(y + e^{-tA^\alpha/2}x)\mathcal{N}_{Q_t}(dy), \quad t > 0, \quad \varphi \in C_b(H),$$

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The Gaussian measure $\mu = \mathcal{N}_Q$ is defined first on the “cylindrical subsets” of H, i.e. subsets of the type

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then it is extended to all Borel sets $\mathcal{B}(H)$ by the Caratheodory teorem.

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For instance, $T_1(t)(B_b(H)) \subset C_b^\infty(H)$ for $t > 0$ while $T_0(t)$ is not strong Feller.

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Besides their own mathematical interests these equations are useful to get informations about the corresponding stochastic PDEs. They arise in several applications, see for instance S. Albeverio and M. Röckner, PTRF 1991 for the Dirichlet forms approach, G. Da Prato and A. Debussche, Journal Math. Pures Appl. 2003, for $3D$ Navier–Stokes equations, L. Zambotti, PTRF 2000, L. Ambrosio, G. Savaré, and L. Zambotti, PTRF 2009, V. Barbu, G. Da Prato, and L. Tubaro, Ann. Probab. 2009, for reflection problems.

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The papers by G. Da Prato, B.Goldys and J. Zabczyk, A. Talarczyk, concern Dirichlet problems in spaces of continuous and bounded functions. Here we consider L^2 spaces.

Associated Sobolev spaces

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Integration formula

$$\int_H \varphi \mathcal{L}_\alpha \psi \, d\mu = -\frac{1}{2}\int_H \langle Q^{(1-\alpha)/2} D\varphi, Q^{(1-\alpha)/2} D\psi\rangle \, d\mu, \quad \varphi, \psi \in \mathcal{E}_\alpha(H).$$

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$W^{1,2}_\alpha(H,\mu) :=$ the completion of $\mathcal{E}_\alpha(H)$ in the norm associated to the scalar product

$$\begin{aligned} \langle \varphi, \psi\rangle_{W^{1,2}_\alpha(H,\mu)} &:= \int_H \varphi\psi \, d\mu + \int_H \langle Q^{(1-\alpha)/2} D\varphi, Q^{(1-\alpha)/2} D\psi\rangle d\mu \\ &= \int_H \varphi\psi \, d\mu + \sum_{k=1}^\infty \int_H \lambda_k^{1-\alpha} D_k\varphi D_k\psi \, d\mu. \end{aligned}$$

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For $\alpha = 0$, $W^{1,2}_0(H,\mu)$ is the domain of the Malliavin derivative in $L^2(H,\mu)$.

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If $K \subset H$ is a Borel set, we define $W^{1,2}_\alpha(K,\mu)$ and $W^{2,2}_\alpha(K,\mu)$ in a similar way.

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$$\begin{cases} \lambda\varphi(x) - \mathcal{L}_\alpha\varphi(x) = f(x), & \text{in } K, \\ \\ \varphi(x) = 0, & \text{on } \partial K. \end{cases}$$

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A weak solution is a function $\varphi \in \mathring{W}_\alpha^{1,2}(K,\mu)$ such that

$$\lambda \int_K \varphi\, v\, d\mu + \frac{1}{2}\int_K \langle Q^{(1-\alpha)/2} D\varphi, Q^{(1-\alpha)/2} Dv\rangle\, d\mu = \int_K f\, v\, d\mu, \quad \forall v \in \mathring{W}_\alpha^{1,2}(K,\mu).$$

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The existence of a unique weak solution follows easily from the Lax-Milgram Lemma.

The operator L_α^K associated to the quadratic form

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- Approximation of $T_\alpha^K(t)$
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- Meaning of the Dirichlet boundary condition
- Regularity of the weak solution: interior regularity, regularity up to the boundary if ∂K is nice

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Proposition. Let either $\alpha = 0$ or $\exists s \in (0,1)$ such that Tr $Q^s < \infty$.

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Let $T_\alpha^\varepsilon(t)$ be the semigroup generated by M_α^ε.

Proposition. Let either $\alpha = 0$ or $\exists s \in (0,1)$ such that Tr $Q^s < \infty$.
Then for every φ in $L^2(K,\mu)$ and $t > 0$, $\lambda > 0$,

$$T_\alpha^K(t)\varphi = \lim_{\varepsilon\to 0} T_\alpha^\varepsilon(t)\widetilde{\varphi}_{|K}, \quad R(\lambda, L_\alpha^K)\varphi = \lim_{\varepsilon\to 0} R(\lambda, M_\alpha^\varepsilon)\widetilde{\varphi}_{|K}$$

Approximation of $T_\alpha^K(t)$

Let L_α be the generator of the O-U semigroup $T_\alpha(t)$ in $L^2(H,\mu)$.
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$$\widetilde{\varphi}(x) = \varphi(x),\ \ x \in K, \quad \widetilde{\varphi}(x) = 0,\ \ x \in K^c.$$

Representation formula for $T_\alpha^K(t)$

Consider the stochastic equation

$$dX_\alpha(t,x) = -\frac{1}{2}Q^{-\alpha}X_\alpha(t,x)dt + Q^{(1-\alpha)/2}dW(t), \quad X(0,x) = x.$$

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where τ_x is the entrance time in the complement of K,

$$\tau_x := \inf\{t \geq 0 : \; X_\alpha(t,x) \in K^c\}, \quad \forall\, x \in K.$$

Proposition. The measure μ is subinvariant for $P_\alpha(t)$, that is,

$$\int_K P_\alpha(t)\varphi\, d\mu \leq \int_K \varphi\, d\mu, \quad t > 0,\ \varphi \in B_b(K),$$

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(Meyer inequalities (1980's) for $\alpha = 0$, Da Prato and Goldys (1991) for $\alpha \in (0,1]$).

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Proof: by localization and by approximation with the operators M_α^ε.

Regular boundaries and traces

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Then the Malliavin surface measure $d\sigma$ is well defined on ∂K.

Integration by parts formula.

$$\int_K D_h\varphi\, d\mu = \frac{1}{\lambda_h}\int_K x_h\varphi\, d\mu + \int_{\partial K} \frac{D_h g}{|Q^{1/2}Dg|}\varphi\, d\sigma, \quad h\in\mathbb{N},\ \varphi\in\mathcal{E}_0(H).$$

Replacing φ by $\lambda_h \varphi^2 D_h g$ and summing over h we get

$$\int_{\partial K} \varphi^2 |Q^{1/2} Dg| \, d\sigma = \int_K \varphi \langle Q^{1/2} D\varphi, Q^{1/2} Dg \rangle \, d\mu + \int_K L_0 g \, \varphi^2 \, d\mu.$$

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So, if $|Q^{1/2}Dg|$ is bounded in K, $|Q^{1/2}Dg| \geq a > 0$ in ∂K and $L_0 g(x)$ has at most linear growth as $|x| \to \infty$, $x \in K$, we have

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and we can define the trace at ∂K of any $\varphi \in W_0^{1,2}(K,\mu)$ (hence, of any $\varphi \in W_\alpha^{1,2}(K,\mu)$) by density, as an element of $L^2(\partial K, \sigma)$.

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Consequence: our weak solutions have null trace at the boundary ∂K.

Regularity up to the boundary

A finite dimensional result. $H = \mathbb{R}^N$ (L.-Metafune-Pallara 2005)

$$\mathcal{L}\varphi = \Delta\varphi - \langle DU, D\varphi\rangle, \quad \mu(dx) = e^{-U(x)}dx.$$

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We have $\mathcal{L} = 2\mathcal{L}_0$ for $U(x) = |x|^2/2$, $Q = I$. If $K = \{x \in \mathbb{R}^N : \ g(x) \leq 1\}$, $\partial U/\partial n + H \leq 0$ at ∂K iff

$$-2L_0 g + \frac{\langle D^2 g \cdot Dg, Dg\rangle}{|Dg|^2} \leq 0 \quad \text{at } \partial K.$$

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Proof: by approximation with finite-dimensional problems.

Examples

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- $K = \{x \in H : \langle x, b\rangle \le 1\}$ halfspace, $g(x) = \langle x, b\rangle$, with $b \neq 0$.

$$h(x) = \frac{1}{|Q^{1/2}b|^2}, \quad x \in H.$$

  h is constant $\Longrightarrow$ maximal Sobolev regularity.

- $K = B(0,1)$, $g(x) = |x|^2$.

$$h(x) = \frac{1 - \mathrm{Tr}Q}{|Q^{1/2}x|^2} + \frac{|Qx|^2}{|Q^{1/2}x|^4} \le \frac{1}{|Q^{1/2}x|^2}\Big(1 - \mathrm{Tr}Q + \lambda_1\Big),$$

  $\lambda_1 =$ maximum eigenvalue of Q. h is upperly bounded if

$$\mathrm{Tr}Q \ge 1 + \lambda_1.$$

  In this case we have maximal Sobolev regularity.

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- About Sobolev regularity up to the boundary: What happens if the function h is unbounded from above on ∂K? Are there counterexamples to $W^{2,2}_{\alpha}$ regularity? (open even in finite dimensions).