American options and integro-differential equations.

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Optimal stopping of Lévy processes

Consider a d-dimensional Lévy process $X = (X_t)_{t \geq 0}$, with characteristic exponent ψ and generating triplet (A, ν, γ), which means

$$\mathbb{E}\left(e^{iz.X_t}\right) = \exp[t\psi(z)], \quad z \in \mathbb{R}^d,$$

where

$$\psi(z) = -\frac{1}{2}z.Az + i\gamma.z + \int \left(e^{iz.x} - 1 - iz.x\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(dx),$$

the matrix $A = (A_{ij})$ is the covariance matrix of the Brownian part, the measure ν on $\mathbb{R}^d \setminus \{0\}$ is the Lévy measure of X, which satisfies $\int (|x|^2 \wedge 1)\nu(dx) < \infty$, and γ is a vector in $\mathbb{R}^d$.

Given a bounded and continuous function f on $\mathbb{R}^d$, we introduce

$$u_f(t,x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(f(x+X_\tau)\right), \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

where $\mathcal{T}_{0,t}$ is the set of all stopping times with values in $[0,t]$. We want to characterize u_f as the unique solution of a variational inequality.

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where $\mathcal{T}_{0,t}$ is the set of all stopping times with values in $[0,t]$. We want to characterize u_f as the unique solution of a variational inequality.
Denote by $\mathcal{L}$ the infinitesimal generator of X. The operator $\mathcal{L}$ can be written as a sum $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is the *local* (differential) part and $\mathcal{B}$ is the *non-local* (integral) part.

For $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\mathcal{A}g(x) = \frac{1}{2}\sum_{i,j=1}^{d} A_{i,j}\frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d}\gamma_i \frac{\partial g}{\partial x_i}(x),$$

and

$$\mathcal{B}g(x) = \int \nu(dy)\left(g(x+y) - g(x) - y.\nabla g(x)\mathbf{1}_{\{|y|\leq 1\}}\right),$$

where ∇g denotes the gradient of g.

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If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, we denote by $\mathcal{D}(\mathcal{O})$ the set of all $\mathcal{C}^\infty$ functions with compact support in $\mathcal{O}$ and by $\mathcal{D}'(\mathcal{O})$ the space of distributions on $\mathcal{O}$. If $u \in \mathcal{D}'(\mathcal{O})$ and $\varphi \in \mathcal{D}(\mathcal{O})$, $\langle u, \varphi \rangle$ denotes the evaluation on the test function φ of the distribution u. Note that if u is a locally integrable function on $\mathcal{O}$,

$$\langle u, \varphi \rangle = \int_{\mathcal{O}} u(x)\varphi(x)dx.$$

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$$\langle u, \varphi \rangle = \int_{\mathcal{O}} u(x)\varphi(x)dx.$$

And the partial derivatives of u are defined by

$$\langle \frac{\partial u}{\partial x_j}, \varphi \rangle = -\int_{\mathcal{O}} u(x)\frac{\partial \varphi}{\partial x_j}(x)dx.$$

Introduce the adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$. For $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$, let

$$\mathcal{B}^*\varphi(x) = \int \left(\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x)\mathbf{1}_{\{|y|\leq 1\}}\right)\nu(dy), \quad x \in \mathbb{R}^d.$$

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If $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function $\mathcal{B}^*\varphi$ is continuous and integrable on $\mathbb{R}^d$, and we have, for $g \in \mathcal{C}_b^2(\mathbb{R}^d)$,

$$\langle \mathcal{B}g, \varphi \rangle = \int_{\mathbb{R}^d} g(x)\mathcal{B}^*\varphi(x)dx.$$

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$$\langle \mathcal{B}g, \varphi \rangle = \int_{\mathbb{R}^d} g(x)\mathcal{B}^*\varphi(x)dx.$$

For $g \in L^\infty(\mathbb{R}^d)$, the distribution $\mathcal{B}g$ can be defined by setting

$$\langle \mathcal{B}g, \varphi \rangle = \int_{\mathbb{R}^d} g(x)\mathcal{B}^*\varphi(x)dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

We can now characterize the value function u_f of an optimal stopping problem with reward function f as the solution of a variational inequality. Note that in the following statement $\partial_t v + \mathcal{L}v$ is to be understood as a distribution.

Theorem

Fix $T > 0$ and let f be a continuous and bounded function on $\mathbb{R}^d$. The function v defined by $v(t,x) = u_f(T-t,x)$ is the only continuous and bounded function on $[0,T] \times \mathbb{R}^d$ satisfying the following conditions:

1. $v(T,.) = f$,
2. $v \geq f$,
3. *On* $(0,T) \times \mathbb{R}^d$, $\partial_t v + \mathcal{L}v \leq 0$,
4. *On the open set* $\{(t,x) \in (0,T) \times \mathbb{R}^d \mid v(t,x) > f(x)\}$, $\partial_t v + \mathcal{L}v = 0$.

Ref. D.L., M. Mikou, 2008. We can summarize properties 2-4 by

$$\max\left(\partial_t v + \mathcal{L}v, f - v\right) = 0.$$

The American put price in an exponential Lévy model

In an exponential Lévy model, the price $(S_t)_{t\in[0,T]}$ of the risky asset is given, under the pricing measure, by

$$S_t = S_0 e^{(r-\delta)t+X_t},$$

where $r > 0$ is the interest rate, $\delta \geq 0$ the dividend rate, and $X = (X_t)_{0\leq t\leq T}$ is a real Lévy process such that $(e^{X_t})_{0\leq t\leq T}$ is a martingale.

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Denote by (σ^2, ν, γ) the generating triplet of X. The martingale property for e^{X_t} is equivalent to the following conditions

$$\int_{\{|x|\geq 1\}} e^x \nu(dx) < \infty$$

and

$$\frac{\sigma^2}{2} + \gamma + \int \left(e^x - 1 - x\mathbf{1}_{\{|x|\leq 1\}}\right) \nu(dx) = 0.$$

The value at time t of an American put with maturity T and strike price K is given by

$$P_t = P(t, S_t),$$

with

$$P(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}\left(e^{-r\tau}\left(K - S^x_\tau\right)_+\right),$$

where $S^x_t = xe^{(r-\delta)t + X_t}$ and $\mathcal{T}_{0,\theta}$ denotes the set of all stopping times with values in $[0, \theta]$.

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3. $P(t,x) \geq (K-x)_+$ and $P(T,x) = (K-x)_+$.
4. $P(t,x) \geq P_e(t,x)$, where
   $P_e(t,x) = \mathbb{E}\left(e^{-r(T-t)}\left(K - S_{T-t}^x\right)_+\right)$.

We assume that one of the following conditions holds

$$\sigma \neq 0, \quad \nu((-\infty, 0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty.$$

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Under this assumption, we have

$$\forall t \in [0, T), \quad \forall x \in [0, +\infty), \quad P_e(t, x) > 0, \text{ therefore } P(t, x) > 0.$$

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We now define the *critical price* at time $t \in [0, T)$ by

$$b(t) = \inf\{x \geq 0 \mid P(t, x) > (K - x)_+\}.$$

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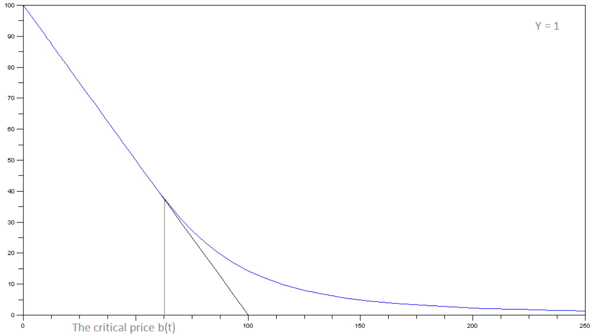
Note that $b(t) \in [0, K)$. It can be proved that $b(t) > 0$.

Graph of $x \mapsto P(t,x)$ for $t < T$



We have

$$\forall t \in [0, T), \quad \forall x \geq 0, \quad (x > b(t) \Leftrightarrow P(t, x) > (K - x)_+).$$

The set

$$C = \{(t, x) \in [0, T) \times [0, +\infty) \mid x > b(t)\}$$

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is called the continuation region.
The critical price is related to the optimal exercise strategy.

$$P(0,x) = \mathbb{E}\left(e^{-r\tau^*}(K - S^x_{\tau^*})_+\right), \text{ with } \tau^* = \inf\{t \in [0,T] \,|\, (t, S^x_t) \notin C\}.$$

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The graph of b is called the *exercise boundary* or *free boundary*.
The function P satisfies the variational inequality

$$\max\left(\partial_t P + \mathcal{L}P - rP, f - P\right) = 0,$$

where $\mathcal{L}$ is the infinitesimal generator of S and $f(x) = (K-x)_+$

The smooth fit property

The continuity of the derivative (with respect to the underlying stock price, across the free boundary) of the American put price is a well known property in the Black-Scholes model, called the *smooth fit* property (see also Zhang (1994) and Bayraktar (2007) for jump-diffusions). In the context of exponential Lévy models, this property may no longer be true.

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In the case of *perpetual* American options, Alili and Kyprianou (2004) proved that a necessary and sufficient condition for smooth fit is that the point 0 is *regular* with respect to the set $(-\infty, 0)$ for the process $\tilde{X}_t := (r-\delta)t + X_t = \log(S_t/S_0)$, which means that $\mathbb{P}(\tau_0 = 0) = 1$, where

$$\tau_0 = \inf\{t > 0 \mid \tilde{X}_t < 0\}.$$

In the case of finite horizon, it can be proved that regularity implies smooth fit (G. Peskir). It follows that the smooth fit property is satisfied by the American put in an exponential Lévy model, if the underlying Lévy process has infinite variation.
Now assume that X has finite variation and let

$$d = r - \delta + \int (e^y - 1)\nu(dy). \quad \text{Note that } \; d = \lim_{t \downarrow 0} \frac{\tilde{X}_t}{t}.$$

Proposition

Consider an exponential Lévy model, in which the generating triplet of the Lévy process is given by (σ^2, ν, γ). Suppose $\sigma^2 = 0$, and $\int (|x| \wedge 1)\nu(dx) < \infty$.

If $d < 0$, smooth fit holds for American put options with finite maturity.

If $r - \delta - \int (e^y - 1)_+ \nu(dy) > 0$, we have, for all $t \in [0, T)$,

$$\partial_x^+ P(t, b(t)) - \partial_x^- P(t, b(t)) \geq \frac{r - \delta - \int (e^y - 1)_+ \nu(dy)}{r - \delta - \int (e^y - 1)\nu(dy)} > 0.$$

The proof of the second part of the proposition relies on the variational inequality. Indeed, in the continuation region, we have

$$d\,x\frac{\partial P}{\partial x}(t,x)+\int\left(P(t,xe^{y})-P(t,x)\right)\nu(dy)-rP(t,x)=-\frac{\partial P}{\partial t}(t,x)\geq 0.$$

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$$d\,x\frac{\partial P}{\partial x}(t,x)\geq\int_{\{y<0\}}\left(P(t,x)-P(t,xe^y)\right)\nu(dy)$$

Note that

$$\lim_{x\to b(t)}\int_{\{y<0\}}\left(P(t,x)-P(t,xe^y)\right)\nu(dy)=b(t)\int(e^y-1)_-\nu(dy).$$

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Note that

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Theorem

If X is a finite variation Lévy process and $d>0$, and if $T>\frac{K}{db^*}$, there exists $t\in[0,T)$ such that

$$\partial_x^+ P(t,b(t))>\partial_x^- P(t,b(t)).$$

Free boundary near maturity

The following result characterizes the limit of the critical price $b(t)$ as t approaches T (see Levendorski (2004), Yang, Jiang and Bian (2006), Bayraktar and Xing (2008), Lamberton and Mikou (2008)).

Theorem

If $\int (e^x - 1)_+ \nu(dx) \leq r - \delta$, *we have*

$$\lim_{t \to T} b(t) = K.$$

If $\int (e^x - 1)_+ \nu(dx) > r - \delta$, *we have* $\lim_{t\to T} b(t) = \xi$, *where* ξ *is the unique real number in the interval* $(0, K)$ *such that* $\varphi(\xi) = rK$, *and* φ *is the function defined by*

$$\varphi(x) = \delta x + \int (xe^y - K)_+ \nu(dy), \quad x \in (0, K).$$

We will concentrate on the case

$$\int (e^x - 1)_+ \nu(dx) < r - \delta, \text{ so that } \lim_{t \to T} b(t) = K.$$

Recall that we then have, if $\nu = 0$ (Black-Scholes case, see Barles et al (1993), Lamberton (1995))

$$\lim_{t \to T} \frac{K - b(t)}{\sqrt{(T-t)|\ln(T-t)|}} = K\sigma.$$

This remains true in the jump-diffusion case (ν finite, see Pham(1997)).

Theorem

If the process X has finite variation and $\int (e^x - 1)_+ \nu(dx) < r - \delta$, we have

$$\lim_{t \to T} \frac{K - b(t)}{T - t} = K \int (e^y - 1)_- \nu(dy) = K \int_{(-\infty,0)} (1 - e^y) \nu(dy).$$

If the process X has infinite variation, we have

$$\lim_{t \to T} \frac{K - b(t)}{T - t} = +\infty.$$

For $t \in [0, T)$, let

$$b_e(t) = \inf\{x \geq 0 \mid P_e(t,x) > K - x\}.$$

In fact, $b_e(t)$ is the unique solution to the equation

$$P_e(t,x) = (K - x)_+.$$

Since $P \geq P_e$, we have $b(t) \leq b_e(t)$. We also have $b_e(t) < K$, because, under our assumptions, $P_e(t,K) > 0$.

The behaviour of b_e

Proposition 1

If the process X has finite variation and $\int (e^x - 1)_+ \nu(dx) < r - \delta$, we have

$$\lim_{t \to T} \frac{K - b_e(t)}{T - t} = K \int (e^y - 1)_- \nu(dy) = K \int_{(-\infty,0)} (1 - e^y) \nu(dy).$$

The behaviour of b_e

Proposition 1

If the process X has finite variation and $\int (e^x - 1)_+ \nu(dx) < r - \delta$, we have

$$\lim_{t \to T} \frac{K - b_e(t)}{T - t} = K \int (e^y - 1)_- \nu(dy) = K \int_{(-\infty,0)} (1 - e^y) \nu(dy).$$

Proof: Denote $\theta = T - t$. The equality $P_e(t, b_e(t)) = K - b_e(t)$ can be written as follows

$$\begin{aligned} K - b_e(t) &= \mathbb{E} e^{-r\theta} \left(K - b_e(t) e^{(r-\delta)\theta + X_\theta} \right)_+ \\ &= K e^{-r\theta} - b_e(t) e^{-\delta\theta} + \mathbb{E} e^{-r\theta} \left(b_e(t) e^{(r-\delta)\theta + X_\theta} - K \right)_+ . \end{aligned}$$

Hence

$$\frac{K}{b_e(t)}\left(1-e^{-r\theta}\right)-\left(1-e^{-\delta\theta}\right)=\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+.$$

Since $\lim_{t\to T} b_e(t)=K$, the left-hand side is equal to $(r-\delta)\theta+o(\theta)$.

Hence

$$\frac{K}{b_e(t)}\left(1-e^{-r\theta}\right)-\left(1-e^{-\delta\theta}\right)=\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+.$$

Since $\lim_{t\to T} b_e(t)=K$, the left-hand side is equal to $(r-\delta)\theta+o(\theta)$. For the study of the right-hand side, let

$$\zeta(\theta)=\frac{K}{b_e(t)}-1 \quad \text{and} \quad \gamma_0=\int(e^y-1)\nu(dy).$$

Hence

$$\frac{K}{b_e(t)}\left(1-e^{-r\theta}\right)-\left(1-e^{-\delta\theta}\right)=\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+ .$$

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$$\zeta(\theta)=\frac{K}{b_e(t)}-1 \quad \text{and} \quad \gamma_0=\int(e^y-1)\nu(dy).$$

Due to the martingale property, $X_t=\gamma_0 t+\sum_{0<0\leq t}\Delta X_s$, so that

$$\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+ \quad = \quad ((r-\delta+\gamma_0)\theta-\zeta(\theta))_+$$

Hence

$$\frac{K}{b_e(t)}\left(1-e^{-r\theta}\right)-\left(1-e^{-\delta\theta}\right)=\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+.$$

Since $\lim_{t\to T} b_e(t)=K$, the left-hand side is equal to $(r-\delta)\theta+o(\theta)$. For the study of the right-hand side, let

$$\zeta(\theta)=\frac{K}{b_e(t)}-1 \quad \text{and} \quad \gamma_0=\int(e^y-1)\nu(dy).$$

Due to the martingale property, $X_t=\gamma_0 t+\sum_{0<0\leq t}\Delta X_s$, so that

$$\begin{aligned}\mathbb{E}e^{-r\theta}\left(e^{(r-\delta)\theta+X_\theta}-\frac{K}{b_e(t)}\right)_+ &= ((r-\delta+\gamma_0)\theta-\zeta(\theta))_+ \\ &\quad +\theta\int\nu(dy)(e^y-1)_++o(\theta).\end{aligned}$$

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$(r-\delta)\theta=((r-\delta+\gamma_0)\theta-\zeta(\theta))_+ +\theta\int\nu(dy)(e^y-1)_+ + o(\theta).$

Estimating the difference $b_e - b$

Proposition 2

If the process X has finite variation and $\int (e^x - 1)_+ \nu(dx) < r - \delta$, we have

$$\lim_{t \to T} \frac{b_e(t) - b(t)}{T - t} = 0.$$

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$$P(t,x) - P_e(t,x) \leq rK \int_0^{T-t} \mathbb{P}\left(x e^{(r-\delta)s + X_s} \leq b(t+s) \right) ds.$$

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2. $$\liminf_{t \to T} \frac{P(t, b_e(t)) - P_e(t, b_e(t))}{b_e(t) - b(t)} > 0.$$

Lower bound for $P(t, b_e(t)) - P_e(t, b_e(t))$

We deduce from the definition of $b_e(t)$ that

$$P(t, b_e(t)) - P_e(t, b_e(t)) = P(t, b_e(t)) - (K - b_e(t))$$

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$$\begin{aligned} P(t, b_e(t)) - P_e(t, b_e(t)) &= P(t, b_e(t)) - (K - b_e(t)) \\ &= P(t, b_e(t)) - P(t, b(t)) - (b(t) - b_e(t)) \\ &\geq (b_e(t) - b(t)) \left(1 + \partial_x^+ P(t, b(t))\right), \end{aligned}$$

where $\partial_x^+ P$ denotes the right-hand derivative of P w.r.t. x. Note that the inequality

$$P(t, b_e(t)) - P(t, b(t)) \geq (b_e(t) - b(t)) \partial_x^+ P(t, b(t))$$

follows from the convexity of $x \mapsto P(t, x)$.

We now deduce a lower bound for $P(t, b_e(t)) - P_e(t, b_e(t))$ from the absence of smooth fit.

Infinite variation

Proposition

If the process X has infinite variation, we have

$$\lim_{t \to T} \frac{K - b_e(t)}{T - t} = +\infty.$$

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We have, as in the proof of Proposition 1,

$$\begin{aligned} (r-\delta)\theta &= \mathbb{E} e^{-r\theta} \left(e^{(r-\delta)\theta + X_\theta} - \frac{K}{b_e(t)} \right)_+ + o(\theta) \\ &= \mathbb{E} \left(e^{(r-\delta)\theta + X_\theta} - \frac{K}{b_e(t)} \right)_+ + o(\theta) \\ &\geq \mathbb{E}(X_\theta + (r-\delta)\theta - \zeta(\theta))_+ + o(\theta), \end{aligned}$$

where we have used the inequality $e^x \geq 1 + x$ and the notation $\zeta(\theta) = \frac{K}{b_e(t)} - 1$.

Hence

$$\limsup_{\theta \to 0} \mathbb{E}\left(\frac{X_\theta}{\theta} + (r-\delta) - \frac{\zeta(\theta)}{\theta}\right)_+ \leq r - \delta.$$

We then deduce from the following Lemma that

$$\lim_{\theta \to 0} \frac{\zeta(\theta)}{\theta} = +\infty.$$

Lemma

If X is a Lévy process with infinite variation, we have, for any real number λ,

$$\lim_{\theta \to 0} \mathbb{E}\left(\frac{X_\theta}{\theta} + \lambda\right)_+ = +\infty.$$

Theorem

Assume

$$\mathbb{E}\left(e^{iuX_t}\right) = \exp\left(t\int\left(e^{iuy} - 1 - iu(e^y - 1)\right)\nu(dy)\right),$$

with $\int(e^y - 1)_+\nu(dy) < r - \delta$ *and, for some* $\kappa > 0$*, and some* $a < 0$*,*

$$\mathbf{1}_{\{a<y<0\}}\nu(dy) = \kappa\frac{1+\varepsilon(y)}{|y|^{1+\alpha}}dy,$$

with $1 < \alpha < 2$ *and* $\lim_{y\to 0}\varepsilon(y) = 0$*.*
Then, we have

$$\lim_{t\to T}\frac{K - b(t)}{(T-t)^{1/\alpha}} = +\infty$$

and

$$\forall\delta > 0, \quad \lim_{t\to T}(T-t)^{\delta}\frac{K - b(t)}{(T-t)^{1/\alpha}} = 0.$$

Conjecture

Under the assumptions of the previous Theorem, we have

$$\lim_{t\to T}\frac{K-b(t)}{(T-t)^{\frac{1}{\alpha}}|\log(T-t)|^{\frac{\alpha-1}{\alpha}}}=C_\alpha K,$$

where

$$C_\alpha=\left(\alpha\kappa\int_0^{+\infty}(e^y-1+y)\frac{dy}{y^{1+\alpha}}\right)^\alpha.$$

Proof of Lemma

Consider the Itô-Lévy decomposition

$$X_t = \gamma t + \sigma B_t + \hat{X}_t + \tilde{X}_t,$$

where

$$\begin{aligned}
\hat{X}_t &= \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|>1\}} = \int_{[0,t]\times\{|x|>1\}} x\, J(ds, dx), \\
\tilde{X}_t &= = \int_{[0,t]\times\{|x|\leq 1\}} x\, \tilde{J}(ds, dx) = \lim_{\varepsilon\downarrow 0} \int_{[0,t]\times\{\varepsilon\leq|x|\leq 1\}} x\, \tilde{J}(ds, dx).
\end{aligned}$$

Here, J is the jump measure of X and

$$\tilde{J}(ds, dx) = J(ds, dx) - ds\nu(dx).$$

Observe that for any two random variables Y and Z we have

$$\begin{aligned}\mathbb{E}(Y+Z)_+ &= \mathbb{E}(Y+Z)\mathbf{1}_{\{Y+Z\geq 0\}}\\ &\geq \mathbb{E}(Y+Z)\mathbf{1}_{\{Y\geq 0\}}\mathbf{1}_{\{Z\geq 0\}}\\ &= \mathbb{E}Y_+\mathbf{1}_{\{Z\geq 0\}} + \mathbb{E}Z_+\mathbf{1}_{\{Y\geq 0\}}.\end{aligned}$$

Using the independence of the elements of the Itô-Lévy decomposition, we deduce

$$\begin{aligned}\mathbb{E}\left(\frac{X_\theta}{\theta}+\lambda\right)_+ &= \mathbb{E}\left(\frac{\gamma\theta+\sigma B_\theta+\hat{X}_\theta+\tilde{X}_\theta}{\theta}+\lambda\right)_+\\ &\geq \mathbb{E}\left(\frac{\sigma B_\theta+\tilde{X}_\theta}{\theta}+\gamma+\lambda\right)_+\mathbb{P}(\hat{X}_\theta\geq 0)\\ &\geq \mathbb{E}\left(\frac{\sigma B_\theta}{\theta}+\frac{\tilde{X}_\theta}{\theta}+\gamma+\lambda\right)_+ e^{-\theta\nu(A)},\end{aligned}$$

where $A=\{x\in\mathbb{R}\mid |x|\geq 1\}$.

Note that B_θ and $\tilde{X}_\theta$ are independent centered random variables so that, using Jensen's inequality,

$$\mathbb{E}\left(\frac{\sigma B_\theta}{\theta}+\frac{\tilde{X}_\theta}{\theta}+\lambda\right)_+ \geq \max\left(\mathbb{E}\left(\frac{\sigma B_\theta}{\theta}+\lambda\right)_+, \mathbb{E}\left(\frac{\tilde{X}_\theta}{\theta}+\lambda\right)_+\right)$$

One can easily check that, if $\sigma \neq 0$, $\lim_{\theta\to 0}\mathbb{E}\left(\frac{\sigma B_\theta}{\theta}+\lambda\right)_+ = +\infty$.

Therefore, it remains to prove that, if $\int_{\{|x|\leq 1\}}|x|\nu(dx) = +\infty$, we have

$$\lim_{\theta\to 0}\mathbb{E}\left(\frac{\tilde{X}_\theta}{\theta}+\lambda\right)_+ = +\infty.$$

Observe that

$$\begin{aligned}\mathbb{E}\left(\frac{\tilde{X}_\theta}{\theta}+\lambda\right)_+ &\geq \mathbb{E}\left(\frac{\tilde{X}^\varepsilon_\theta}{\theta}+\lambda\right)_+\\ &\geq \mathbb{E}\left(\frac{\tilde{X}^\varepsilon_\theta}{\theta}\right)_+ - |\lambda|,\end{aligned}$$

where

$$\begin{aligned}\tilde{X}^\varepsilon_\theta &= \int_{[0,\theta]\times\{\varepsilon\leq|x|\leq 1\}} x\,\tilde{J}(ds,dx)\\ &= \bar{X}^\varepsilon_\theta - c_\varepsilon\theta,\end{aligned}$$

where

$$\bar{X}^\varepsilon_\theta = \int_{[0,\theta]\times\{\varepsilon\leq|x|\leq 1\}} x\,J(ds,dx) = \sum_{0<s\leq\theta}\Delta X_s \mathbf{1}_{\{\varepsilon\leq\Delta X_s\leq 1\}}$$

and

$$c_\varepsilon = \int_{\{\varepsilon\leq|y|\leq 1\}} y\nu(dy).$$

Note that

$$\begin{aligned} \mathbb{E}\left(\tilde{X}^{\varepsilon}_{\theta}\right)_{+} &= \mathbb{E}\left(\bar{X}^{\varepsilon}_{\theta} - c_{\varepsilon}\theta\right)_{+} \\ &= \theta(-c_{\varepsilon})_{+} + \theta\int_{\{\varepsilon\le|y|\le 1\}} y_{+}\nu(dy) + o(\theta). \end{aligned}$$

Hence

$$\liminf_{\theta\to 0}\mathbb{E}\left(\frac{X_{\theta}}{\theta}\right)_{+} \ge (-c_{\varepsilon})_{+} + \int_{\{\varepsilon\le|y|\le 1\}} y_{+}\nu(dy).$$

Clearly, if $\int_{\{0<|y|\le 1\}} y_{+}\nu(dy) = +\infty$, we get

$$\lim_{\theta\to 0}\mathbb{E}\left(\frac{\tilde{X}_{\theta}}{\theta}+\lambda\right)_{+} = +\infty.$$

On the other hand, if $\int_{\{0<|y|\le 1\}} y_{+}\nu(dy) < +\infty$, the condition $\int_{\{0<|y|\le 1\}} |y|\nu(dy) = +\infty$ implies that $\lim_{\varepsilon\to 0} c_{\varepsilon} = -\infty$, so that the same conclusion follows.