Kolmogorov equations in physics and finance

Modena (Italy)

V.D. Konakov (Moscow)
Discrete “parametrix” method and its applications

8 -10 September 2010

PARAMETRIX FOR DIFFUSIONS.

We describe briefly the parametrix method proposed by Mc Kean and Singer (1967) which gives an infinite series representation for the transition density of the diffusion governed by the following SDE

$$dY(t) = m(Y)dt + \Lambda(Y)dW(t), Y(0) = x, t \in [0,1]. \qquad (1)$$

For simplicity we consider the time homogeneous case. In contrast to the E.Levy method this method is well suited for approximations. The most precise results are obtained under the non degeneracy assumption of the diffuson matrix Λ and we suppose, first, that this matrix is non degenerate. For $0 < s < 1$ and $(x,y) \in R^{2p}$ we define diffusions $\widetilde{Y} = \widetilde{Y}_{s,x,y}$ that are defined for $s \leq t \leq 1$ by $\widetilde{Y}(s) = x$ and

$$d\widetilde{Y}(t) = m(y)dt + \Lambda(y)dW(t).$$

PARAMETRIX FOR DIFFUSIONS

The infinite series representation for the transition density p(s, t, x, y) of the diffusion (1) has the following form

$$p(s,t,x,y) = \widetilde{p}(s,t,x,y) + \sum_{r=1}^{\infty} \widetilde{p} \otimes H^{(r)}(s,t,x,y), \tag{2}$$

where the convolution type binary operation $\otimes$ is defined by

$$f \otimes g(s,t,x,y) = \int_s^t du \int_{R^d} f(s,u,x,z) g(u,t,z,y) dz$$

and the kernel

$$H(s,t,x,y) = (L - \widetilde{L})\widetilde{p}(s,t,x,y),$$

$$\begin{aligned} L &= \frac{1}{2}\sum_{ij} \sigma_{ij}(x)\partial^2_{x_i x_j} + \sum_i m_i(x)\partial_{x_i}, \\ \widetilde{L} &= \frac{1}{2}\sum_{ij} \sigma_{ij}(y)\partial^2_{x_i x_j} + \sum_i m_i(y)\partial_{x_i}, \end{aligned}$$

Here $\Sigma(x) = \|\sigma_{ij}(x)\| = \Lambda\Lambda^*(x), H^{(r)} = H^{(r-1)} \otimes H.$

DISCRETE PARAMETRIX

We explain now how to derive a finite series expansion of the transition density $p_n(\frac{j}{n}, \frac{i}{n}, x, y)$ of the Markov chain X_n in the model

$$X_n\left(\frac{k+1}{n}\right) = X_n\left(\frac{k}{n}\right) + \frac{1}{n} m\left(X_n\left(\frac{k}{n}\right)\right) + \frac{1}{\sqrt{n}}\varepsilon_n\left(\frac{k+1}{n}\right), k = 0, ..., n-1 \tag{3}$$

$p_n(\frac{j}{n}, \frac{i}{n}, x, y)$ denotes the conditional density of $X_n\left(\frac{i}{n}\right)$ at the point y, given $X_n(\frac{j}{n}) = x$. Again we apply the parametrix method and for this purpose we introduce additional "frozen" Markov chains. These are defined as follows. For all $0 \leq \frac{j}{n} \leq 1$ and $(x, y) \in R^{2p}$ we define the Markov chains $\widetilde{X}_{\frac{j}{n},x,y}$. For fixed j, x and y, the chain is defined for i with $j \leq i \leq n$. The dynamics of the chain is given by

$$\widetilde{X}_n\left(\frac{k+1}{n}\right) = \widetilde{X}_n\left(\frac{k}{n}\right) + \frac{1}{n} m(y) + \frac{1}{\sqrt{n}}\widetilde{\varepsilon}_n\left(\frac{k+1}{n}\right), k = 0, ..., n-1 \tag{4}$$

DISCRETE PARAMETRIX

The stochastic structure of the R^p- valued innovations $\widetilde{\varepsilon}_n\left(\frac{i}{n}\right)$ is described as follows. Given $\widetilde{X}_n\left(\frac{l}{n}\right)=x_l$ for $l=j,...,i$ the variable $\widetilde{\varepsilon}_n\left(\frac{i}{n}\right)$ has a conditional density $q\left(y,\cdot\right)$. Note that the conditional distribution of $\widetilde{X}_n\left(\frac{i+1}{n}\right)-\widetilde{X}_n\left(\frac{i}{n}\right)$ does not depend on the "past" $\widetilde{X}_n\left(\frac{l}{n}\right)$ for $l=j,...,i$. Let us call $\widetilde{X}_n$ *the Markov chain frozen at y*. We put $\widetilde{Y}_n\left(t\right)=\widetilde{X}_n\left(\left[nt\right]\right)$ and we write $\widetilde{p}_n(\frac{j}{n},\frac{i}{n},x,y)$ for the conditional density of $\widetilde{X}_{\frac{j}{n},x,y}$ at the point y, given $\widetilde{X}_{\frac{j}{n},x,y}(\frac{j}{n})=x$. Note that, as in the case of a "frozen" diffusion, the variable y acts here twice: as the argument of the density and as a defining quantity of the process $\widetilde{X}_{\frac{j}{n},x,y}$. For $0\leq\frac{j}{n}<\frac{i}{n}\leq1$ the finite series representation of the density $p_n(\frac{j}{n},\frac{i}{n},x,y)$ (a discrete analog of (2) has the following form

$$p_n(\frac{j}{n},\frac{i}{n},x,y)=\widetilde{p}_n(\frac{j}{n},\frac{i}{n},x,y)+\sum_{r=1}^{i-j}\widetilde{p}_n\otimes_n H_n^{(r)}(\frac{j}{n},\frac{i}{n},x,y), \qquad (5)$$

DISCRETE PARAMETRIX

where the convolution type binary operation $\otimes_n$ is defined by

$$f \otimes_n g(\frac{j}{n}, \frac{i}{n}, x, y) = \sum_{k=j}^{i-1} \frac{1}{n} \int f\left(\frac{j}{n}, \frac{k}{n}, x, z\right) g(\frac{k}{n}, \frac{i}{n}, z, y) dz,$$

$$\begin{aligned}
H_n(\frac{k}{n}, \frac{i}{n}, x, y) &= \left(L_n - \widetilde{L}_n\right) \widetilde{p}_n(\frac{k}{n}, \frac{i}{n}, x, y), \\
L_n f(\frac{k}{n}, \frac{i}{n}, x, y) &= n \int p_n(\frac{k}{n}, \frac{k+1}{n}, x, z) \left[f(\frac{k+1}{n}, \frac{i}{n}, z, y) - f(\frac{k+1}{n}, \frac{i}{n}, x, y)\right], \\
\widetilde{L}_n f(\frac{k}{n}, \frac{i}{n}, x, y) &= n \int \widetilde{p}_n^y(\frac{k}{n}, \frac{k+1}{n}, x, z) \left[f(\frac{k+1}{n}, \frac{i}{n}, z, y) - f(\frac{k+1}{n}, \frac{i}{n}, x, y)\right].
\end{aligned}$$

DISCRETE PARAMETRIX

The proof of (5) is based on simple telescopic arguments. Namely, we have a simple identity

$$p_n(\frac{j}{n},\frac{i}{n},x,y)-\widetilde{p}_n(\frac{j}{n},\frac{i}{n},x,y)=\sum_{k=j}^{i-1}\frac{1}{n}\int p_n(\frac{j}{n},\frac{k}{n},x,z)\times$$

$$n\int\left[p_n(\frac{k}{n},\frac{k+1}{n},z,z')-\widetilde{p}_n(\frac{k}{n},\frac{k+1}{n},z,z')\right]\widetilde{p}_n(\frac{k+1}{n},\frac{i}{n},z',y)dzdz'.$$

Note that the close telescopic arguments were used later by [JKMP, 2005] and [Guyon, 2006] to study the difference for the Euler scheme $\Delta_t^n=P_t^n-P_t$, where $P_tf(x)=Ef(X_t^x), P_t^nf(x)=Ef(X_t^{n,x})$ and the telescopic identity for this case has the following form

$$P_t^n-P_t=\sum_{k=0}^{[nt]-1}P_{\frac{k}{n}}^n\Delta_{\frac{1}{n}}^nP_{t-\frac{k+1}{n}}+P_{[nt]/n}^n\Delta_{t-[nt]/n}.$$

ACCURACY OF APPROXIMATIONS

The main idea consists in a careful comparison of the two parametrix series, namely, (2) and (5).

1. **Nondegenerate diffusion matrix. Convergence rate estimate [KM, 2000].**

   Theorem. *Under appropriate regularity conditions the following estimate holds*

   $$\sup_{(x,y)\in R^{2p}} \left(1 + \|y - x\|^{S'}\right) |p_n(0,1,x,y) - p(0,1,x,y)| = O\left(n^{-1/2}\right).$$

   The positive integer S' is, roughly speaking, the numver of moments of the innovations ε_n in the Markov chain model (3).

2. **Nondegenerate diffusion matrix. Edgeworth type expansions [KM, 2009] .** The sequence of Markov chains is defined on the grid $\{0, h, 2h, ..., nh = T\}$, and we allow $T \to 0$ but not too fast, namely, we suppose that there exists $\kappa < 1/5$ such that

$$\liminf_{n\to\infty} Tn^{\kappa} > 0.$$

Theorem. *Under stronger regularity conditions there exists a constant $\delta > 0$ such that the following expansion holds*

$$\sup_{(x,y)\in R^{2p}} \left[T^{d/2} \left(1 + \left\| \frac{y-x}{\sqrt{T}} \right\|^{S'} \right) |p_n(0,T,x,y) - p(0,T,x,y) \right.$$

$$\left. - h^{1/2}\pi_1(0,T,x,y) - h\pi_2(0,T,x,y) \Big| \right] = O\left(h^{1+\delta}\right)$$

where

$$\pi_1(0,T,x,y) = (p \otimes \mathcal{F}_1[p])(0,T,x,y),$$

$$\pi_2(0,T,x,y) = (p \otimes \mathcal{F}_2[p])(0,T,x,y) + p \otimes \mathcal{F}_1[p \otimes \mathcal{F}_1[p]](0,T,x,y)$$
$$+\frac{1}{2} p \otimes \left(L_*^2 - L^2\right) p(0,T,x,y).$$

$$\mathcal{F}_1[f](s,t,x,y) = \sum_{|\nu|=3} \frac{\chi_\nu(y)}{\nu!} D_x^\nu f(s,t,x,y),$$
$$\mathcal{F}_2[f](s,t,x,y) = \sum_{|\nu|=4} \frac{\chi_\nu(y)}{\nu!} D_x^\nu f(s,t,x,y),$$

and $\chi_\nu(y)$ *is the* $\nu-$*th cumulant of the density* $q(y,\cdot)$ *of the innovations* ε_n.The operator L_* is defined as $\widetilde{L}$ but with the coefficients "frozen" at the point x.

3. **Degenerate diffusion matrix. Kolmogorov type equations [KMM, 2009,2010].**

We consider $R^p \times R^p$ - valued diffusion process that follows the dynamicst

$$\begin{cases} dX_t = b(X_t, Y_t)dt + \sigma(X_t, Y_t)dW_t, X_0 = x, \\ dY_t = X_t dt, Y_0 = y. \end{cases} \tag{6}$$

where $(W_t)_{t\geq 0}$ is a standard $p-$dimensional Brownian motion defined on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathrm{P}\right)$ satisfying the usual assumptions. We assume that σ is uniformly elliptic and that b, σ are bounded, Lipschitz continuous functions so that there exists a unique strong solution to (6).

Remark 1 *Note that the above result would remain valid if we replaced the dynamics of Y_t in (6) by $Y_t = y + \int_0^t F(X_s)ds$ for a $C^{2+\alpha}$, $\alpha > 0$, Lipschitz continuous mapping $F : R^p \to R^p$ s.t. $\nabla F \nabla F^*$ is non degenerated, i.e. $\exists c_0, \forall (\xi, x) \in R^p \times R^p \, |\langle \nabla F \nabla F^* (x) \xi, \xi \rangle| \geq c_0 |\xi|^2$ .Indeed, in such a case, $\left(\overline{X}_s, \overline{Y}_s\right)_{s\in[0,T]} \doteqdot (F(X_s), Y_s)$ follows a dynamics of type (6).*

ACCURACY OF APPROXIMATIONS

We develop analogously to the contnuous case a parametrix approach to express the density of the Markov chain in term of the density of an auxiliary frozen random walk. The random walk is degenerated as well, but has a density after a su¢cient number of time steps. The local limit theorem is then derived from an accurate comparison of the parametrix expansions of the densities of the process and the chain. To motivate this result we can consider the case of the approximation of a "digital Asian call" i.e. of the quantity

$$\mathbf{P}\left[\left(T^{-1}Y_T - X_T\right)^+ > K\right] \text{ for a given } K \in R^+.$$

Indeed, the local limit theorem associated to our scheme directly relates the densities of the discrete and continuous objects which is not the case if we only consider a discretization of the non degenerate component and a numerical estimation of the integral, since in that case the approximating couple can fail to have a density.

ACCURACY OF APPROXIMATIONS

Parametrix expansion for the transition density of the diffusion.

For all $0 \leq t \leq T$, $((x,y),(x',y')) \in (R^{2p})^2$

$$p(t,(x,y),(x',y')) = \widetilde{p}(t,(x,y),(x',y')) + \sum_{r=1}^{\infty} \widetilde{p} \otimes H^{(r)}(t,(x,y),(x',y')), \tag{7}$$

where $\widetilde{p}(t,(x,y),(x',y'))$ is the transition density of the "frozen" diffusion

$$\begin{aligned} d\widetilde{X}_s^{t,x',y'} &= \sigma(x', y' - x'(t-s))\, dW_s + b(x',y')ds, \widetilde{X}_0^{t,x',y'} = x, \quad (8) \\ d\widetilde{Y}_s^{t,x',y'} &= \widetilde{X}_s^{t,x',y'} ds, \widetilde{Y}_0^{t,x',y'} = y. \end{aligned}$$

The key point is that the above process is gaussian. Note that the second argument in σ depends on s. In fact, to obtain correct estimates by parametrix method we have to transport the terminal condition y' by the backward flow θ_t^{-1} corresponding to the solution of the deterministic ODE $\frac{d}{ds}\theta_s = x'ds, \theta_t = y'$.The following estimate can be derived from the parametrix series (7)

$$p(t,(x,y),(x',y')) \leq C\widehat{p}_c(t,(x,y),(x',y')),$$

where $\widehat{p}_c(t,(x,y),(x',y'))$ is the Gaussian density with a well-known "Kolmogorov type" singularity in time

$$\widehat{p}_c(t,(x,y),(x',y')) \doteq$$

$$\frac{c^p 3^{p/2}}{(2\pi t^2)^p} \times \exp\left(-c\left[\frac{|x'-x|^2}{4t} + 3\frac{\left|y'-y-\frac{(x'+x)t}{2}\right|^2}{t^3}\right]\right).$$

4. Model and results.

Now, fix $T > 0$, $N_0 \in N^*$ and let $h_0 = T/N_0$ be the "micro" time discretization step. Let $n \in N^*$ be large enough so that the natural "frozen" chain associated to (8) has a density, and define the "macro" scale time step $h = nh_0$ and set $N = N_0/n$ the total number of "macro" time steps over $[0, T]$. For all $i \in [0, N]$ set $t_i \doteqdot ih$. For any $(x, y) \in R^{2p}$, we define on the time grid $\{0, ..., t_N\}$ an R^{2p} valued Markov chain $\left(Z^h_{t_i}\right)_{i\in[0,N]}$ $= \left(\left(X^h_{t_i}, Y^h_{t_i}\right)\right)_{i\in[0,N]}$ whose dynamics is given by

$$Z^h_0 = (x, y)^*, \quad \text{and } \forall i \in [0, N-1]$$

$$\left\{ \begin{array}{c} X^h_{t_{i+1}} = X^h_{t_i} + b\left(Z^h_{t_i}\right)h + \sigma\left(Z^h_{t_i}\right)\sqrt{h}\eta^1_{i+1} \\ Y^h_{t_{i+1}} = Y^h_{t_i} + \left(X^h_{t_i} + \frac{\gamma_n}{2}b\left(Z^h_{t_i}\right)h + \sigma\left(Z^h_{t_i}\right)\sqrt{h}\eta^2_{i+1}\right)h, \end{array} \right. , \qquad (9)$$

where $\gamma_n \doteqdot 1 + \frac{1}{n}$. The variables $(\vartheta_i)_{i\in(0,N]} \doteqdot \left(\eta^1_i, \eta^2_i\right)_{i\in(0,N]}$ are i.i.d. centered $2p-$dimensional r.v. such that the variable ϑ_i has density $q_n(\eta_1, \eta_2)$ satisfying appropriate moment conditions.

ACCURACY OF APPROXIMATIONS

The "frozen" counterpart of (9) is of the following form. For fixed (x,y), (x',y') $\in R^{2p}$, $j\in(0,N]$ we

define $\left(\widetilde{Z}^h_{t_i}\right)_{i\in(0,j]}$ by

$$\widetilde{Z}^h_0 = (x,y)^*, \quad \text{and } \forall i \in [0,j]$$

$$\begin{cases} \widetilde{X}^h_{t_{i+1}} = \widetilde{X}^h_{t_i} + b(x',y')h + \sigma(x', y' - x'(t_j - t_i))\sqrt{h}\widetilde{\eta}^1_{i+1} \\ \widetilde{Y}^h_{t_{i+1}} = \widetilde{Y}^h_{t_i} + \left\{\widetilde{X}^h_{t_i} + \frac{\gamma_n}{2} b(x',y')h + \sigma(x', y' - x'(t_j - t_i))\sqrt{h}\widetilde{\eta}^2_{i+1}\right\} h, \end{cases}$$

The i.i.d. variables $\left(\widetilde{\eta}^1_i, \widetilde{\eta}^2_i\right)_{i\in(0,N]}$ have density $q_n(\eta_1,\eta_2)$. Careful application of the parametrix method allows to obtain the following local limit theorem for densities for the degenerate diffusion.

ACCURACY OF APPROXIMATIONS

Theorem 2 *There exists $c > 0$ such that*

$$\sup_{(x,y),(x',y')\in R^{2p}} \left[\chi_{\sqrt{T}}\left(x'-x, y'-y-T\times\frac{x+x'}{2}\right)+\right.$$

$$\left.(1+|x|+|x'|)\sup_{\delta\in[0,1]}\widehat{p}_c\left(T(1+\delta),(x,y),(x'y')\right)\right]^{-1}\times$$

$$|p_h(T,(x,y,),(x',y'))-p(T,(x,y,),(x',y'))| = O(h^{-1/2}),$$

where

$$\chi(u,v) = \left(1+(|u|^2+|v|^2)^S\right)^{-1}, \chi_\rho(u,v) = \rho^{-4p}\chi\left(\frac{u}{\rho},\frac{v}{\rho^3}\right).$$

Note from the above result that the bigger is S, the better is the control on the tails.

TWO DIMENTIONAL RAYLEIGH MODEL

The following problem which was first posed in 1905 by Karl Pearson in the following terms: "A man starts from a point O and walks l yards in a straight line; he then turns hrough any angle whatever and walks l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at distance between r and $r + dr$ from his starting point O: " Mark Pinsky gives a solution of Pearson's problem allowing that the length of each step is a random variable. This model is called *Rayleigh model of random flight.* In probabilistic terms the displacement after n steps is the random variable

$$S_n = X_1 + X_2 + ... + X_n$$

where $[X_n : n \geq 1]$ is a sequence of independent and identically distributed R^2 - valued random variables whose distribution is given in polar coordinates r, θ by the density $(2\pi)^{-1} f(r)\, dr d\theta$ where $f(r) = a^{-1} \exp(-r/a)$, $a > 0$. If ξ is a random variable with the radial density function $f(r)$ then $E\xi = a$, $Var\xi = a^2$.

TWO DIMENTIONAL RAYLEIGH MODEL

Thus we consider *"the random walk over spheres in* R^2 *"* It's easy to see that the sequence S_n is a Markov chain whose one-step transition probability density $p(x, y)$ is given by the formula

$$p(x, y) = \Delta^{-1}\Psi\left(\frac{y-x}{\Delta}\right), \Delta = a^2, \Psi(u) = \frac{1}{2\pi\|u\|}\exp(-\|u\|), \|u\| = \sqrt{u_1^2 + u_2^2}.$$

GENERALISATION

For $x \in R^p$ we consider a matrix field $\Lambda(x)$, where $\Lambda(x)$ is a symmetric and positively definite $p \times p$ matrix, and a vector field $m(x) = (m_1(x), ..., m_p(x))$. We start from a point x_0. For the unit ellipsoid

$$S_{\circ}(1) = \left\langle \Sigma^{-1}(x_0)(x - x_0), x - x_0 \right\rangle = 1,$$

where $\Sigma(z) = \|\sigma_{ij}(z)\| = \Lambda^2(z)$, there exists a unique measure $\mu = \mu_{S_0(1)}$ which is rotationally invariant with respect to the orthogonal group $O(p)$ and normalized in such a way that $\int_{S_0(1)} \mu(d\xi) = 1$, which we shall call the normalized Lebesgue measure (in fact this is the Haar measure which is the image of the normalized Lebesgue measure on the unit sphere $B_0(1) = \{x : \|x - x_0\| = 1\}$ under linear transformation $y = \Lambda(x_0)x$. Then we choose a μ -distributed point ξ on the unit ellipsoid $S_{\circ}(1)$ and a random point η on the ray with the directional unit vector $\mathbf{e}$ having the origin at x_0 and passing through ξ. The random point η is supposed to be chosen independently of ξ and has a radial density $f(r)$. Let r be a coordinate of a point. We consider a point $x_0 + r\mathbf{e}$ on a random ellipsoid $S_0(r)$ centered at x_0 and then we add a nonrandom shift $x_0 + r\mathbf{e}$ $+m(x_0)\Delta$, where $\Delta \to 0$ and may depend on the $f(r)$.Denote a resulting point by x_1. Then we make the same step starting from x_1, etc.

GENERALISATION

As a result we get "the random walk over ellipsoids with drift". Using the well-known formula

$$\int_{S_0(r)} \exp\left(i\langle t, \xi\rangle\right) \mu_{S_0(r)}(d\xi) = 2^{\frac{p-2}{2}} \Gamma\left(\frac{p}{2}\right) \frac{J_{\frac{p-2}{2}}\left(\left|\Lambda\left(x_0\right) t\right| r\right)}{\left(\left|\Lambda\left(x_0\right) t\right| r\right)^{\frac{p-2}{2}}},$$

where $J_\nu(x)$ is the Bessel function of the order ν, we obtain for the Fourier transform of the one-step transition density

$$2^{\frac{p-2}{2}} \Gamma\left(\frac{p}{2}\right) \int_0^\infty \frac{J_{\frac{p-2}{2}}\left(\left|\Lambda\left(x_0\right) t\right| r\right)}{\left(\left|\Lambda\left(x_0\right) t\right| r\right)^{\frac{p-2}{2}}} f(r)\, dr, |t| \neq 0.$$

To obtain the transition density $p(x, y)$ it remains to apply the Fourier Inversion Theorem.

GENERALISATION

Examples.

1. We put $m(x) \equiv 0$ and consider

$$f(r) = \frac{1}{\Gamma(p)} a^{-p} r^{p-1} \exp\left(-\frac{r}{a}\right)$$

as a radial density. For this case

$$p(x,y) = \Delta^{-p/2} q\left(x, \frac{y-x}{\sqrt{\Delta}}\right),$$

where $\Delta = (p+1)\, a^2$

$$q(x,z) = \frac{(p+1)^{p/2}}{2^p \pi^{(p-1)/2} \Gamma\left(\frac{p+1}{2}\right) \det \Lambda(x)} \exp\left\{-\left|\Lambda^{-1}(x) z\right| \sqrt{p+1}\right\}.$$

It is easy to check that

$$\int q(x,z)\, z_i dz = 0, \int q(x,z)\, z_i z_j dz = \sigma_{ij}(x).$$

For the general case $m(x) \neq 0$ we have the following formula

$$p(x,y) = \Delta^{-p/2} q\left(x, \frac{y-x-\Delta m(x)}{\sqrt{\Delta}}\right)$$

GENERALISATION

2. For the radial density

$$f(r) = C_p a^{-p} r^{p-1} \exp\left(-r^2/a^2\right)$$

and for the case $m(x) \neq 0$ we have the following formula

$$p(x, y) = \Delta^{-p/2} q\left(x, \frac{y - x - \Delta m(x)}{\sqrt{\Delta}}\right),$$

where

$$q(x, z) = \phi(z; \mathbf{0}, \Sigma(x)), \Delta = a^2/2.$$

If we put $a = \sqrt{\frac{2}{n}}$ then $\Delta = \frac{1}{n}$ and the transition density of our random walk after n steps is equal to the transition density $p(n, 0, 1)$ from 0 to 1 in the model

$$X_n\left(\frac{k+1}{n}\right) = X_n\left(\frac{k}{n}\right) + \frac{1}{n} m\left\{X_n\left(\frac{k}{n}\right)\right\} + \frac{1}{\sqrt{n}} \varepsilon_n\left(\frac{k+1}{n}\right), X(0) = x_0. \quad (10)$$

where the conditional density of the innovation $\varepsilon_n\left(\frac{k+1}{n}\right)$ given the past $X_n\left(\frac{k}{n}\right)$, $X_n\left(\frac{k-1}{n}\right)$ ... depends only on the last value $X_n\left(\frac{k}{n}\right)$. Given $X_n\left(\frac{i}{n}\right) = x_i$ for $i = 0, ..., k$, the variable $\varepsilon_n\left(\frac{k+1}{n}\right)$ has a conditional density $q(x_k, \cdot)$. We see that (10) is in fact the Euler scheme for stochastic differential equation

$$dY(t) = m(Y_t)\, dt + \Lambda(Y_t)\, dW_t, \; Y_t(0) = x_0. \quad (11)$$

GENERALISATION

Under the conditions (A1)-(A4) of our main theorem we get that the $n-$step transition density $p(n,x,y)$ of our random walk over ellipsoids admits the following approximation

$$\sup_{(x,y)\in R^{2p}} \left(1+\|y-x\|^{2(S'-1)}\right)$$

$$\times \left|p(n,x,y)-p(0,1,x,y)-n^{-1}\pi_2(x,y)\right| = O(n^{-1-\delta})$$

for some $\delta > 0$ and S' defined in Assumption (A2). Here $p(0,1,x,y)$ is the transition density of the difusion (11) and

$$\pi_2(x,y) = \frac{1}{2} p \otimes (L_*^2 - L^2) p(0,1,x,y).$$