Weak solution for forward-backward stochastic systems

Giuseppina Guatteri

Politecnico di Milano

Modena, 8-10 September 2010

Joint work with F. Delarue

Université de Nice Sophia-Antipolis

Plan of the talk

A Quick introduction to Backward Stochastic Differential Equation (BSDE) and Forward-Backward Stochastic Differential Equations (FBSDE)

B Setting of the problem and statement of the result

C Existence

D Uniqueness in law

E Comments

A.1 BSDE

Fix $T > 0$, $n > 0$ integer.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, $(B_t)_{t \in [0,T]}$ is a Brownian Motion in $\mathbb{R}^n$ defined on that space and $(\mathcal{F}_t)_{t \in [0,T]}$ its filtration completed.

Let $f : [0,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ an assigned function, the so-called *generator* and η an assigned random variable $\mathcal{F}_T$-measurable.

We seek for a **couple** of processes (Y, Z) both **adapted** to $\mathcal{F}_t$ such that

$$Y_t = \eta + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \qquad t \in [0,T]$$

Thanks to the measurability condition the two processes are related by:

$$\langle Y_\cdot, B_\cdot \rangle_{[0,t]} = \int_0^t Z_s\, ds$$

First existence-uniqueness result in the non-linear case: [Pardoux & Peng '90]

A.2 Forward-Backward System

Fix $d > 0$ integer.
Let $x \in \mathbb{R}^d$ and $b, \sigma : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{d\times n} \to \mathbb{R}^d, \mathbb{R}^{d\times n}$, and $G : \Omega \times \mathbb{R}^d \to \mathbb{R}^n$.

We seek for a triplet of processes (X, Y, Z) **adapted** to $\mathcal{F}_t$ such that:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)\, dB_s \\ \\ Y_t = G(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dB_s, \qquad\qquad t \in [0, T] \end{cases}$$

when the forward equation depends on (Y, Z) it is called **fully coupled Forward-Backward system**.

A.3 Applications

- Finance: e.g. Options pricing, Large Investors;

- Stochastic optimal control: stochastic maximum principle, stochastic H.J.B. equations in case when the state equation has random coefficients, stochastic optimal control in infinite dimensions.

When the forward equation is decoupled and all the coefficients are deterministic, the associated *semilinear* Kolmogorov equation is:

$$\begin{cases} \partial_t u(t,x) + Lu(t,x) = f(t,x,u(t,x),\sigma^*(x)\nabla_x u(t,x)), \quad x \in \mathbb{R}^d \\ u(T,x) = G(x) \end{cases}$$

where $L\psi(x) = \frac{1}{2} tr[\sigma\sigma^*(\cdot)\nabla^2\psi(\cdot)](x) + \langle b(x), \nabla\psi(x)\rangle$.

A.4 Well posedeness of Forward-Backward System

Assume smooth coefficients

- if b,σ are independent of (Y, Z) then standard **fixed point technique** apply;

- if b,σ depend on the backward unknowns, different pictures:
  - solution may not exist in any time intervall,
  - there may be infinite solutions,
  - **if σ is independent of** Z one can prove existence in small time intervall [Antonelli ,'90]

A.5 Solution on time interval of arbitrary lengh

Assume from now on that σ is independent of Z .

There are two families of methods for getting unique solvability in $[0,T]$, T arbitrarly chosen:

1. to work directly on the system using **monotonicity** conditions and exponential norms to perform **fixed point technique**, [Hu& Peng, Peng & Wu, Yong, Pardoux & Tang]:

   $\uparrow$ stochastic coefficients, $\downarrow$ ugly conditions for the coefficients to verify

2. to use the **connection** between the stochastic system and the associated non-linear PDE, [Ma & Protter& Yong, Hu& Peng, Delarue, G.& Lunardi]

   $\uparrow$ fit well non-linear case, easy conditions for the coefficients to verify, $\downarrow$ no stochastic coefficients, almost always σ invertible

In any case all the coefficients appearing in the for-bac system are at **least Lipschitz continuous** and **at most of linear growth** in all the variables

B.1 Setting of the problem

Assume that σ is invertible.

For any $(t,x) \in [0,T] \times \mathbb{R}^d$ reformulate the for-bac system as follows:

$$(E)\quad \begin{cases} \forall s \in [t,T], \\ X_s = x + \int_t^s b(r, X_r, Y_r, Z_r) dr + \int_t^s \sigma(r, X_r, Y_r) dB_r, \\ Y_s = G(X_T) + \int_s^T f(r, X_r, Y_r, Z_r) dr - \int_s^T Z_r^* \sigma(r, X_r, Y_r) dB_r. \end{cases}$$

The associated Kolmogorov equation is the quasilinear PDE, set $a(t,x,y) = \sigma\sigma^*(t,x,y)$:

$$(\mathcal{E})\quad \begin{cases} \partial_t u(t,x) + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}\big(t,x,u(t,x)\big) \partial^2_{x_i,x_j} u(t,x) \\ + \sum_{i=1}^d b_i\big(t,x,u(t,x),\nabla_x u(t,x)\big) \partial_{x_i} u(t,x) \\ + f\big(t,x,u(t,x),\nabla_x u(t,x)\big) = 0,\ (t,x) \in [0,T[\times \mathbb{R}^d, \\ u(T,x) = G(x),\ x \in \mathbb{R}^d. \end{cases}$$

B.2 Identifications

Under suitable assumptions, having a solution $(X_t^{t,x}, Y_t^{t,x}, Z_t^{t,x})$, one can get the solution u of $(\mathcal{E})$ as follows:

$$\mathbb{E}(Y_t^{t,x}) = u(t,x)$$

On the other side having a *regular enough solution* u of $(\mathcal{E})$ and a solution $X_t^{t,x}$ of:

$$X_t = x + \int_0^t b\big(s, X_s, u(s, X_s), \nabla_x u(t,x)\big)ds + \int_0^t \sigma\big(s, X_s, u(s, X_s)\big)dB_s.$$

one can prove that:

$$(X_t^{t,x}, u(t, X_t^{t,x}), \nabla_x u(s, X_s))$$

is a solution of system (E)

B.3 Strong vs Weak

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let (B) be a Brownian Motion $(B_s)_{0\leq s\leq T}$ with natural filtration completed $\mathcal{F}_t$. Then a **strong solution** (X, Y, Z) of (E), is a triplet of processes (X, Y, Z) adapted to $\mathcal{F}_t$ such that (E) is fullfilled $\mathbb{P}$ almost-surely.

While a **weak solution** of (E), denoted by $((\Omega, \{\mathcal{G}\}, \mathbb{P}, B), (X, Y, Z))$ is:
a filtered probability space $(\Omega, (\mathcal{G}_s)_{0\leq s\leq T}, \mathbb{P})$;
a Brownian Motion $(B_s)_{0\leq s\leq T}$ defined on the above space;
a triplet (X, Y, Z) of adapted processes to $\mathcal{G}_t$, such that (E) is fullfilled $\mathbb{P}$ almost-surely.

In the notion of weak solution the probability space **is not fixed a priori**, the triplet (X, Y, Z) is **not** necessarily adapted to the filtration generated by the noise.

two notions of uniqueness

- **pathwise (strong)**;
- **in law (weak)**: $(B, X, Y, Z)(\mathbb{P}) \sim (\tilde{B}, \tilde{X}, \tilde{Y}, \tilde{Z})(\mathbb{P})$.

B.4 Assumption (A)

We assume that there exist five constants $\alpha_0 > 0$, H, K, $\lambda > 0$ and Λ, such that, for all $t \in [0, T]$, $(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, $(x', y', z') \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$,

(A.1) $|(b, \sigma, G)(t, x, y, z)| \leq \Lambda(1+|y|+|z|), \ |f(t, x, y, z)| \leq \Lambda(1+|y|+|z|^2)$.

(A.2) $\forall \zeta \in \mathbb{R}^d, \ \langle \zeta, a(t, x, y)\zeta \rangle \geq \lambda |\zeta|^2$, where $a(t, x, y) = \sigma\sigma^*(t, x, y)$.

(A.3)
$$\left[\begin{array}{l} |a(t,x,y) - a(t,x,y')| \leq K\,|y - y'|, \\ |b(t,x,y,z) - b(t,x,y',z')| \leq K(|y - y'| + |z - z'|), \\ |f(t,x,y,z) - f(t,x,y',z')| \leq K(1 + |z| + |z'|)(|y - y'| + |z - z'|). \end{array} \right.$$

(A.4) $|a(t, x', y) - a(t, x, y)| + |G(x') - G(x)| \leq H\,|x' - x|^{\alpha_0}$.

No regularity assumptions with respect to the x variable for b, f.

B.5 Main result

Theorem 1 *Let* $(t,x) \in [0,T] \times \mathbb{R}^d$*. Then, under Assumption* **(A)***, the Forward-Backward SDE* (E) *admits a weak solution* $((\Omega, \{\mathcal{F}\}, \mathbb{P}, B), (X, Y, Z))$ *with initial condition* (t,x)*.*

Moreover, if $((\tilde{\Omega}, \{\tilde{\mathcal{F}}\}, \tilde{\mathbb{P}}, \tilde{B}), (\tilde{X}, \tilde{Y}, \tilde{Z}))$ *denotes another weak solution with initial condition* (t,x)*, then the distributions* $(\tilde{B}, \tilde{X}, \tilde{Y}, \tilde{Z})(\tilde{\mathbb{P}})$ *and* $(B, X, Y, Z)(\mathbb{P})$ *on the space* $\mathcal{C}([t,T], \mathbb{R}^d) \times \mathcal{C}([t,T], \mathbb{R}^d) \times \mathcal{C}([t,T], \mathbb{R}) \times L^2([t,T], \mathbb{R}^d)$ *are equal.*

From an analytical point of view, there exists a unique solution to the PDE $(\mathcal{E})$ *in the space :*

$$\mathcal{V} \equiv \Big\{ u \in \ \mathcal{C}^0([0,T] \times \mathbb{R}^d, \mathbb{R}) \cap \mathcal{C}^{0,1}([0,T[\times \mathbb{R}^d, \mathbb{R}) \cap W^{1,2,d+1}_{\mathrm{loc}}([0,T[\times \mathbb{R}^d, \mathbb{R}),$$
$$\exists\ \gamma > 0, \sup_{(t,x) \in [0,T[\times \mathbb{R}^d} \Big(|u(t,x)| + (T-t)^{1/2-\gamma} |\nabla_x u(t,x)| \Big) < +\infty \Big\},$$

with $W^{1,2,d+1}_{\mathrm{loc}}([0,T[\times \mathbb{R}^d, \mathbb{R}) \equiv \{u : [0,T[\times \mathbb{R}^d \to \mathbb{R}, |u|,\ |\nabla_x u|,\ |\nabla^2_{x,x} u|,\ |\partial_t u| \in L^{d+1}_{\mathrm{loc}}([0,T[\times \mathbb{R}^d, \mathbb{R})\}$*. The process* (Y,Z) *can then be chosen to satisfy:*

$$\forall s \in [t,T],\ Y_s = u(s, X_s),\ \forall s \in [t,T[,\ Z_s = \nabla_x u(s, X_s).$$

C.1 Existence of a solution of the Kolmogorov equation associated $(\mathcal{E})$

- regularizing procedure for the coefficients $\to$ we get a family of regular solutions u_n for the associated regularized problem $(\mathcal{E}_n)$;

- a-priori estimate on appropriate modulus of continuity for u_n;

- compactness argument $\to$ extract a solution u in the space $\mathcal{V}$.

difficulty: we can not control pointwise $\nabla^2_{x,x} u_n$, due to the lack of regularity of the coefficients Schauder theory do no apply

solution: local integral estimate Let $p \geq 1$. There exist a constant $\alpha \in]0,1]$, depending only on costants appearing in **(A)** and T (and not on p), and a constant $C(p)$, depending only on costants appearing in **(A)**, p and T, such that, for all $R \geq 1$, $\delta \in]0,T]$, $z \in \mathbb{R}^d$,

$$\int_{T-\delta}^{T} \int_{B(z,R)} \big[(T-s)^{1-\alpha}\big(|\partial_t u(s,y)| + |\nabla^2_{x,x} u(s,y)|\big)\big]^p ds\, dy \leq C(p)\delta R^d,$$

C.2 Existence of a weak solution for (E)

step 1 Consider first the following forward SDE (F)

$$X_t = x + \int_0^t b\big(s, X_s, u(s, X_s), \nabla_x u(s, X_s)\big) ds + \int_0^t \sigma\big(s, X_s, u(s, X_s)\big) dB_s.$$

difficulty: σ only Hölder continuous (**no strong solution** counter-example by **Barlow**), drift only locally bounded

solution: formulate it as Martingale problem, exploiting the fact that σ is invertible and continuous, we have well posedness of the Martingale problem $\rightarrow$ **there exists a unique (in law) weak solution** $((\Omega, \{\mathcal{F}\}, \mathbb{P}, B), X)$

C.3 Existence of a weak solution for (E)

step 2 Turn to the backward component of the system and define (Y, Z), for all $t \in [0, T]$, by $Y_t \equiv u(t, X_t)$, $Z_t \equiv \nabla_x u(t, X_t)$ and verify, by Itô formula, that fullfill the backward equation

difficulty: the solution u is not regular enough to perform classical Itô formula

solution: we have applied the so-called Itô-Krylov formula and a localization procedure

D.1 Uniqueness in law for (E): a decoupling strategy

Let $(\tilde{\Omega}, \{\tilde{\mathcal{F}}\}, \tilde{\mathbb{P}}, \tilde{B}, (U, V, W))$ be another solution the FBSDE (E)

Set

$$\bar{V}_t \equiv u(t, U_t), \ \forall t \in [0, T[, \ \bar{W}_t \equiv \nabla_x u(t, U_t)$$

identify $(\bar{V}, \bar{W})$ **with** (V, W) $\rightarrow$ **the forward component** of the system (E) **becomes** SDE (F) satisfied by X

Thus we can derive uniqueness in law of the solution from the weak uniqueness property of (F).

D.2 After a series of unfortunate events...

Difficulties

- f is quadratic in z;
- unbounded terms do not allow to apply Itô-Krylov formula;
- the estimate on $\nabla^2 u$ is just local and integral

Solutions

- introduce a quadratic function ϕ and evaluate $d_t\phi(|V_t - \bar{V}_t|^2)$;
- perform an appropriate change of probability space, using Girsanov transformation;
- apply Krylov and Berstein estimates and use a discrete version of Gronwall's Lemma

we eventually prove the uniqueness in law for the weak solution

D.3 Uniqueness of the solution for the PDE

Let $\tilde{u} \in \mathcal{V}$ be another solution to $(\mathcal{E})$ then for all $(t,x) \in [0,T] \times \mathbb{R}^d$, there exists a weak solution to (E) $((\tilde{\Omega}, \{\tilde{\mathcal{F}}\}, \tilde{\mathbb{P}}, \tilde{B}), (\tilde{X}, \tilde{Y}, \tilde{Z}))$, where $\tilde{Y} \equiv \tilde{u}(\cdot, \tilde{X}_\cdot)$ and $\tilde{Z} \equiv \nabla \tilde{u}(\cdot, \tilde{X}_\cdot)))$.

Hence

the weak uniqueness property for (E) yields

$$u(t,x) = \mathbb{E}(Y_t^{t,x}) = \tilde{\mathbb{E}}(\tilde{Y}_t^{t,x}) = \tilde{u}(t,x)$$

E Comments

- *Strong Solvability of* (E). The for-bac system (E) is not strongly solvable since the forward equation reduces to a SDE with Hölder continuous coefficients.

  If the coefficient σ is assumed to be continuous in (t, x) and Lipschitz continuous with respect to the variable x, the SDE (F) turns out to be **strongly solvable** then, the solution built would be **strong**.

  The method to get **uniqueness still applies** and permits to establish uniqueness in the pathwise sense;

- the notion of weak solution, for for-bac systems, has been introduced in [Ma & Antonelli], recently also a weaker notion of solution has been introduced [Ma& Zhang& Zheng]

- if f would be at most of linear growth in z multidimensional extention should be possible;

- it should also be possible to weaken the Lipschitz hypothesis on f in y to a monotonicity assumptions

Short bibliography:

- **Pardoux & Peng, [Systems Control Lett.,(14), 1990.]**
- **Antonelli, [Annals of Applied Probability,(3), 1993.]**
- **Peng & Wu, [SIAM J. Control Optim.,(37), 1999.]**
- **Yong, [PTRF,(107), 1997.]**
- **Hu & Peng, [PTRF,(103), 1995.]**
- **Pardoux & Tang,[PTRF,(114), 1999]**
- **Ma & Protter &Yong, [PTRF,(98), 1994]**
- **Delarue,[Stoc.Proc.Appl.,(2), 2002]**
- **Barlow,[J. London Math. Soc.,(26), 1982]**
- **Antonelli & Ma[Stoc. Anal. and Appl.,(21), 2003]**
- **Ma & Zhang &Zheng [Ann. Probab.(36,n.6), 2008]**