American-style options, stochastic volatility, and degenerate parabolic variational inequalities

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Kolmogorov Equations in Physics and Finance
Based on joint work with P. Daskalopoulos and C. Pop

Collaborators

Joint work with

- Panagiota Daskalopoulos, *Columbia University*, and

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- Panagiota Daskalopoulos, *Columbia University*, and
- Camelia Pop, Ph.D. student, *Rutgers University*.

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- In particular, we stochastic volatility processes, such as the Heston process, and their generalizations.
- We consider their Kolmogorov PDEs and initial/boundary value and obstacle problems arising in option pricing.
- We explore questions of existence, uniqueness, and regularity of solutions to variational inequalities, as well as the regularity and geometric properties of the free boundary.

Some difficulties characteristic of option-pricing problems

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- Processes defined by stochastic differential equations with
  - Degenerate diffusion coefficients.
  - Non-Lipschitz coefficients.
- Payoff or obstacle functions which are at most Lipschitz.
- Discontinuous data for initial/boundary value problems.
- Other complications include:
  - Unbounded domains.
  - Unbounded coefficients.
  - Ubounded boundary data and obstacle functions.
  - Non-local obstacle constraints.

A motivating example

These complications arise in simple, low-dimensional examples:

- Find the American-style down-and-out put option value function when the underlying asset price is a Heston stochastic volatility process.

We often restrict to this example in our presentation, though many of our results admit natural generalizations.

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- Bayraktar & Xing (2009), Bayraktar, Kardaras & Xing (2010), Ekström & Tysk (2010), Nyström (2008)

Processes defined by stochastic differential equations with non-Lipschitz coefficients

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- Berkaoui, Bossy, & Diop (2008), Höpfner (2010)

Degenerate elliptic/parabolic partial differential equations

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- Daskalopoulos & Hamilton (1998), Koch (1999)
- Bayraktar, Xing & Kardaras (2009, 2010), Ekström & Tysk (2010), Frentz & Nyström (2010)

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- Chadam & Chen (2006, 2008, 2010), Ekström (2004), Laurence & Salsa (2009), Nyström (2008)

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Degenerate stationary/evolutionary variational inequalities

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- ...and undoubtedly others.

Variational problems with discontinuous boundary data

Motivations from finance are well-known. For example:

- A European-style up-and-out call option, with strike $K = e^k$ and upper barrier $B = e^b$, defines an initial/boundary value problem with discontinuous data at $(T, b) \in \partial_p Q$ if $b > k$:
  - $u(T, x) = (e^x - e^k)^+$, when $-\infty < x < b$;
  - $u(t, b) = 0$, when $0 \leq t < T$.

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- A European-style up-and-out call option, with strike $K = e^k$ and upper barrier $B = e^b$, defines an initial/boundary value problem with discontinuous data at $(T, b) \in \partial_p Q$ if $b > k$:
  - $u(T, x) = (e^x - e^k)^+$, when $-\infty < x < b$;
  - $u(t, b) = 0$, when $0 \leq t < T$.
- An American-style down-and-out put option, strike $K = e^k$ and lower barrier $B = e^b$, defines an initial/boundary value problem with discontinuous data at $(T, b) \in \partial_p Q$ if $b < k$:
  - $u(T, x) = (e^k - e^x)^+$, when $b < x < \infty$;
  - $u(t, b) = 0$, when $0 \leq t < T$.

Heston's stochastic volatility process

The process proposed by Heston (1993) is defined by $S(u) = \exp(X(u))$, where

$$dX(u) = (r - Y(u)/2))\,du + \sqrt{Y(u)}\,dW_1(u), \quad X(t) = x,$$
$$dY(u) = \kappa(\theta - Y(u))\,du + \sigma\sqrt{Y(u)}\,dW_2(u), \quad Y(t) = y,$$

where $(W_1(u), dW_3(u))$ is two-dimensional Brownian motion, $W_2(u) := \rho W_1(u) + \sqrt{1-\rho^2}\,W_3(u)$, κ, θ, σ are positive constants, $\rho \in (-1, 1)$, and $r \geq 0$.

Heston's parabolic PDE

Denote $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$, let $\mathscr{O} \subset \mathbb{R}^2_+$ be a domain, and let $Q := [0, T) \times \mathscr{O}$. Given $h : \mathscr{O} \to \mathbb{R}$, define

$$u(t, x, y) := e^{-r(T-t)} \mathbb{E}^{t,x,y}_{\mathbb{Q}} \left[h(X(T), Y(T)) \right],$$

and note that

$$-u' + Au = 0 \quad \text{on } Q, \quad u(T, \cdot) = h \quad \text{on } \mathscr{O},$$

where

$$-Au := \frac{y}{2}\left(u_{xx} + 2\rho\sigma u_{xy} + \sigma^2 u_{yy}\right) + (r - y/2)u_x + \kappa(\theta - y)u_y - ru.$$

Degenerate elliptic/parabolic PDEs

Suppose $(t,x) \in Q = [0,T) \times \mathscr{O}$ and $\mathscr{O} \subset \mathbb{R}^n$, and

$$-Au(t,x) := \frac{1}{2}\sum_{i,j} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_i b_i(t,x)\frac{\partial u}{\partial x_i}(t,x) - c(t,x)u(t,x).$$

If $\xi^T A(t,x)\xi \geq \mu(t,x)|\xi|^2$, $\xi \in \mathbb{R}^n$, where $\mu(x) > 0$, then A is *elliptic (parabolic) on* Q if $\mu > 0$ on Q, and A is *uniformly elliptic (parabolic) on* Q if $\mu \geq \delta$ on Q, for some constant $\delta > 0$. This condition fails for the Heston operator, as $\mu = 0$ along $\{y = 0\}$ component of $\bar{\mathscr{O}}$ and the operator is "degenerate".

Weighted Sobolev spaces

Definition

We need a weight function when defining our Sobolev spaces,

$$\mathfrak{w}(x,y) := \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x|-\mu y}, \quad \beta = \frac{2\kappa\theta}{\sigma^2}, \mu = \frac{2\kappa}{\sigma^2},$$

for $(x,y) \in \mathscr{O}$ and a suitable positive constant, γ. Then

$$H^1(\mathscr{O},\mathfrak{w}) := \{u \in L^2(\mathscr{O},\mathfrak{w}) : (1+y)^{1/2} u \in L^2(\mathscr{O},\mathfrak{w}), \text{ and } y^{1/2} Du \in L^2(\mathscr{O},\mathfrak{w})\},$$

where

$$\|u\|^2_{H^1(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} y\left(u_x^2 + u_y^2\right) \mathfrak{w}\, dxdy + \int_{\mathscr{O}} (1+y) u^2 \,\mathfrak{w}\, dxdy.$$

Weighted Sobolev spaces and the boundary at $y = 0$

Let $H^1_0(\mathscr{O},\mathfrak{w})$ be the closure in $H^1(\mathscr{O},\mathfrak{w})$ of $C^\infty_0(\mathscr{O})$.

Definition

Denote $\Gamma_0 := \bar{\mathscr{O}} \cap (\mathbb{R} \times \{0\})$ and let $H^1_0(\mathscr{O} \cup \Gamma_0,\mathfrak{w})$ be the closure in $H^1(\mathscr{O},\mathfrak{w})$ of $C^\infty_0(\mathscr{O} \cup \Gamma_0)$.

One has the following useful result:

Lemma

If $\beta > 1$, then $H^1_0(\mathscr{O} \cup \Gamma_0,\mathfrak{w}) = H^1_0(\mathscr{O},\mathfrak{w})$.

A bilinear form

Definition

The Heston generator, $-A$, may be written in divergence form and defines a bilinear map, $a : V \times V \to \mathbb{R}$, via

$$a(u, v) := (Au, v)_H, \quad u, v \in C_0^\infty(\mathscr{O}).$$

where $V := H_0^1(\mathscr{O}, \mathfrak{w})$ and $H := L^2(\mathscr{O}, \mathfrak{w})$.

Gårding inequality

Proposition

Let $r, \sigma, \kappa, \theta \in \mathbb{R}$ be constants such that

$$\beta := \frac{2\kappa\theta}{\sigma^2} > 0, \quad \sigma \neq 0, \quad \textit{and} \quad -1 < \rho < 1.$$

Then, there are positive constants, C_1, C_2, depending at most on the coefficients $r, \kappa, \theta, \rho, \sigma$, such that for all $u \in V$,

$$a(u,u) \geq \frac{1}{2} C_2 \|u\|_V^2 - C_3 \|(1+y)^{1/2} u\|_H^2.$$

Continuity estimate

Proposition

There is a positive constant, C_1, depending at most on the coefficients $r, \kappa, \theta, \rho, \sigma$ such that

$$|a(u,v)| \leq C_1 \|u\|_V \|v\|_V, \quad \forall (u,v) \in V \times V.$$

Variational inequality problem

Problem

Let $f \in L^2(\mathscr{O}, \mathfrak{w})$ and $g, \psi \in H^1(\mathscr{O}, \mathfrak{w})$ such that $\psi \le g$ on $\mathscr{O}$, and suppose $\beta \neq 1$. Find $u \in K$ such that

$$a(u, v-u) \ge (f, v-u)_{L^2(\mathscr{O},\mathfrak{w})}, \text{ with } u = g \text{ on } \partial_\beta \mathscr{O},$$
$$\forall v \in K \text{ with } v = g \text{ on } \partial \mathscr{O}_\beta,$$

where $K := \{v \in H^1(\mathscr{O}, \mathfrak{w}) : v \ge \psi\}$.

Remarks

Remark

- Here $\partial_\beta\mathscr{O} := \partial\mathscr{O} - \Gamma_0$, $\beta > 1$, and $\partial_\beta\mathscr{O} := \partial\mathscr{O}$, $\beta < 1$, where we recall that $\Gamma_0 = \bar{\mathscr{O}} \cap (\mathbb{R} \times \{0\})$.
- Recall that $H^1_0(\mathscr{O} \cup \Gamma_0, \mathfrak{w}) = H^1_0(\mathscr{O}, \mathfrak{w})$ when $\beta > 1$. Hence, for $\beta \neq 1$, the boundary conditions for u, v are given by

$$u - g, v - g \in H^1_0(\mathscr{O}, \mathfrak{w}).$$

Existence and uniqueness of solutions

Theorem

There exists a unique solution to the stationary variational inequality for the Heston generator.

The result is proved by the penalization method and adapting the arguments of Bensoussan and Lions (1982).

Another weighted Sobolev space

Definition

Let

$$H^2(\mathscr{O},\mathfrak{w}) := \{u \in L^2(\mathscr{O},\mathfrak{w}) : (1+y)^{1/2}u, y^{1/2}Du, yD^2u \in L^2(\mathscr{O},\mathfrak{w})\},$$

where

$$\|u\|^2_{H^2(\mathscr{O},\mathfrak{w})} := \int_{\mathscr{O}} \left[y^2\left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right) \right. \\ \left. + y\left(u_x^2 + u_y^2\right) + (1+y)u^2\right] \mathfrak{w}\, dxdy.$$

Let $H^2_{\text{loc}}(\mathscr{O},\mathfrak{w})$ denote the space of functions $u \in H^2(\mathscr{O}',\mathfrak{w})$ for all $\mathscr{O}' \Subset \mathscr{O}$.

H^2 regularity and solution to the strong problem

Suppose $\psi(x,y) = (K - e^x)^+$ or $(e^x - K)^+$. Then,

Theorem

If u is the solution to the stationary variational inequality for the Heston generator, then $u \in H^2(\mathscr{O}, \mathfrak{w})$ and

$$Au - f \geq 0, \quad u - \psi \geq 0, \quad (Au - f)(u - \psi) = 0 \text{ on } \mathscr{O}.$$

Remarks

Remark

- The preceding result is proved by adapting the arguments of Bensoussan and Lions (1982) and Jaillet, Lamberton, and Lapeyre (1990).
- We expect, by work in progress, that $u \in W^{2,p}(\mathscr{O}, \mathfrak{w})$, $p > 2$, and $u \in C^{1,1}(\mathscr{O})$.

A failure of coercivity

- Simple attempts to adapt the argument Bensoussan and Lions (1982) in their proof existence and uniqueness of solutions to the "strong" variational inequality to the Heston generator, $-A$, fail because the bilinear map defined by A is *non-coercive*.

A change of dependent variable

- To circumvent the lack of coerciveness, we employ the change of dependent variable

$$\tilde{u}(t,x,y) = e^{-\lambda(1+y)(T-t)}u(t,x,y), \quad u \in V, (t,x,y) \in Q,$$

by analogy with the familiar *exponential shift* change of dependent variable $\tilde{u} = e^{-\lambda(T-t)}u$.

Transformation of the equations

- One finds that the *non-coercive* parabolic problem,

$$-u' + Au = f \text{ on } Q, \quad u(T) = h \text{ on } \mathscr{O}, \quad u = g \text{ on } \Sigma,$$

is transformed, for $t \in [T - \delta, T]$ and sufficiently small δ, into an equivalent *coercive* parabolic problem,

$$-\tilde{u}' + \tilde{A}\tilde{u} = \tilde{f} \text{ on } Q, \quad \tilde{u}(T) = h \text{ on } \mathscr{O}, \quad \tilde{u} = \tilde{g} \text{ on } \Sigma,$$

- An obstacle condition $u \geq \psi$ is transformed into an equivalent obstacle condition $\tilde{u} \geq \tilde{\psi}$.

Transformation of the bilinear form

The bilinear form on $V \times V$ (defined by the weight $\mathfrak{w}$) associated to the operator $\tilde{A}(t)$ (with suitable boundary conditions) is

$$\tilde{a}(t; \tilde{u}(t), v) := (\tilde{A}(t)\tilde{u}(t), v)_{L^2(\mathscr{O},\mathfrak{w})}. \tag{1}$$

We then obtain the key continuity estimate and Gårding inequality for $\tilde{a}(t)$.

Continuity estimate and Gårding inequality for the transformed Heston operator

Proposition

For a sufficiently large positive constant λ, depending only the coefficients of A, and a sufficiently small positive constant $\delta < T$, depending only on λ and the coefficients of A, the bilinear map $\tilde{a}(t) : V \times V \to \mathbb{R}$ obeys

$$|\tilde{a}(t; u, v)| \leq C\|u\|_V\|v\|_V,$$
$$\tilde{a}(t; v, v) \geq \frac{\alpha}{2}\|v\|_V^2,$$

for all $u, v \in V$ and $t \in [T - \delta, T]$.

Change of Sobolev weight and transformation back to original problem

The weight in our previous definition of weighted Sobolev spaces,

$$\mathfrak{w}(x,y) := \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x|-\mu y}, \quad (x,y) \in \mathscr{O},$$

is replaced, when transforming back from a solution $\tilde{u}$ to a solution u to the original problem, by

$$\begin{aligned} \tilde{\mathfrak{w}}(x,y) &:= e^{-2\lambda M(1+y)} \mathfrak{w}(x,y) \\ &= \frac{2}{\sigma^2} y^{\beta-1} e^{-\gamma|x|-\mu y-2\delta\lambda(1+y)}, \quad (x,y) \in \mathscr{O}, \end{aligned}$$

where $M > T$ is a constant.

Setup for the evolutionary problem

Definition

- Recall that $V = H^1_0(\mathscr{O}, \mathfrak{w})$ and $H = L^2(\mathscr{O}, \mathfrak{w})$. Denote $\mathscr{V} := L^2(0, T; V)$, $\mathscr{V}' := L^2(0, T; \mathscr{V}')$, $\mathscr{H} := L^2(0, T; H)$, and $\mathscr{K} := \{v \in \mathscr{V} : v \geq \psi\}$, given $\psi \in \mathscr{V}$.
- The transformed Heston generator, $-A(t)$, defines a linear map $\mathscr{A}(t) \in \mathscr{L}(V, V')$ and a bilinear map $a(t) : V \times V \to \mathbb{R}$ by
$$a(t; u, v) := \mathscr{A}(t)u(v), \quad u, v \in V,$$
and
$$\mathscr{A}(t)u(v) := (A(t)u, v)_H, \quad u, v \in C(0, T; C_0^\infty(\mathscr{O})).$$

Boundary conditions

Definition

- Given $u, g \in L^2(0, T; H^1(\mathscr{O}, \mathfrak{w}))$, then $u = g$ on $\partial_\beta \mathscr{O} \times [0, T)$, $\beta \neq 1$, means that $u - g \in L^2(0, T; H^1_0(\mathscr{O}, \mathfrak{w}))$.
- As usual, we may assume, without loss, that $g = 0$, and the condition $\psi \leq g$ on $[0, T) \times \partial\mathscr{O}$ is replaced by $\psi \leq 0$ on $[0, T) \times \partial\mathscr{O}$.
- Recall that $\partial_\beta \mathscr{O} := \partial\mathscr{O} - \Gamma_0$, $\beta > 1$, and $\partial_\beta \mathscr{O} := \partial\mathscr{O}$, $\beta < 1$, where $\Gamma_0 = \bar{\mathscr{O}} \cap (\mathbb{R} \times \{0\})$.

Evolutionary variational inequality problem

Problem

Suppose $f, \psi \in \mathscr{H}$ and $h \in V$ with $h \geq \psi(T, \cdot)$ on $\mathscr{O}$ and $\psi \leq 0$ on $[0, T) \times \partial\mathscr{O}$. Find $u \in \mathscr{K}$, with $u' \in \mathscr{H}$, such that

$$-(u'(t), v - u(t))_H + a(t; u(t), v - u(t)) \geq (f(t), v - u(t))_H,$$
$$\forall v \in V \text{ with } v \leq \psi(t, \cdot), \quad t \in [0, T].$$

Existence and uniqueness of solutions

Theorem

There exists a unique solution to the evolutionary variational inequality for the Heston generator.

As with the stationary variational inequality, the result is proved by the penalization method and adapting the arguments of Bensoussan and Lions (1982).

Regularity for solutions to the strong problem for the parabolic *Heston* variational inequality

Suppose $\psi(t, x, y) = (e^x - K)^+$ or $(K - e^x)^+$, $(t, x, y \in Q$. Using our weighted Sobolev spaces and estimates, we adapt the Bensoussan-Lions regularity theory to establish

Theorem

If u is the solution to the evolutionary variational inequality for the Heston generator, then $u \in L^2(0, T; H^2(\mathscr{O}, \mathfrak{w}))$.

Given this regularity, a solution to the strong problem for the parabolic Heston variational inequality is a solution to the more familiar "complementarity" or strong form of the obstacle problem.

Solution to the strong form of the obstacle problem

Theorem

Given $f \in L^2(0,T;L^2(\mathscr{O},\mathfrak{w}))$, $g \in L^2(0,T;H^1(\mathscr{O},\mathfrak{w}))$, and $h \in H^1(\mathscr{O},\mathfrak{w})$ obeying $g \geq \psi$ on Q, $h \geq \psi$ on $\mathscr{O}$, there is a unique $u \in L^2(0,T;H^2(\mathscr{O},\mathfrak{w}))$ solving

$$\begin{aligned} -u' + Au &\geq f \text{ on } Q, \\ u &\geq \psi \text{ on } Q, \\ (-u' + Au - f)(u - \psi) &= 0 \text{ on } Q, \\ u &= g \text{ on } \partial_\beta \mathscr{O} \times [0,T), \\ u(T) &= h \text{ on } \mathscr{O}. \end{aligned}$$

Improved regularity

Remark

We expect, by work in progress, that $u \in L^2(0, T; W^{2,p}(\mathscr{O}, \mathfrak{w}))$, $p > 2$, and $u \in C^{1,1}(Q)$.

Stochastic representations and their consequences

- Given a stochastic representation of a solution, certain regularity and geometric properties follow readily.

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- The following stochastic representations may be derived by adapting methods of Bensoussan & Lions (1982), Friedman (1976), and Øksendal (2003).

Probabilistic solutions to stationary variational equalities I

Problem (Stationary variational equality)

Let $\mathscr{O} \subset \mathbb{R} \times (0, \infty)$ be a domain with C^2 boundary, $\partial\mathscr{O}$, let $f \in C(\mathscr{O})$ obey

$$|f(x,y)| \leq C_1(1+y), \quad (x,y) \in \mathscr{O},$$

and let $g \in C_b(\partial\mathscr{O})$. Find $u \in C^2(\mathscr{O}) \cap C(\bar{\mathscr{O}})$ such that

$$\begin{aligned} Au &= f \quad \text{on } \mathscr{O}, \\ u &= g \quad \text{on } \partial_\beta\mathscr{O}. \end{aligned}$$

Probabilistic solutions to stationary variational equalities II

Theorem (Uniqueness of solutions to Problem 4.1)

Let u be a solution to Problem 4.1. Then $u = u^*$ on $\bar{\mathscr{O}}$, with

$$\begin{aligned} u^*(x,y) := \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau} g(X(\tau),Y(\tau)) 1_{\{\tau<\infty\}}\right] \\ + \mathbb{E}_{\mathbb{Q}}\left[\int_0^\tau e^{-rs} f(X(s),Y(s))\,ds\right], \quad (x,y) \in \mathscr{O}, \end{aligned} \tag{2}$$

where $r > 0$ and τ is the exit time from $\mathscr{O}$ of the process $(X(s),Y(s))_{s\geq 0}$ starting at $(x,y) \in \mathscr{O}$.

Probabilistic solutions to stationary var'l inequalities I

Problem (Stationary variational inequality)

Let $\mathscr{O}, f, g$ be as in Problem 4.1 and let $\psi \in C(\bar{\mathscr{O}})$ obey $\psi \leq g$ on $\partial_\beta \mathscr{O}$, and

$$|f(x,y)| \leq C_1(1+y) \text{ and } |\psi(x,y)| \leq C_2(1+e^{C_3 x}), \quad (x,y) \in \mathscr{O}.$$

Find $u \in C^2(\mathscr{O}) \cap C(\bar{\mathscr{O}})$ such that

$$\begin{aligned} Au \geq f \text{ and } u \geq \psi \text{ on } \mathscr{O}, \\ (Au - f)(u - \psi) = 0 \text{ on } \mathscr{O}, \\ u = g \text{ on } \partial_\beta \mathscr{O}. \end{aligned}$$

Probabilistic solutions to stationary var'l inequalities II

Theorem (Uniqueness of solutions to Problem 4.3)

Let u be a solution to Problem 4.3. Then $u = u^*$ on $\bar{\mathscr{O}}$, with

$$
\begin{aligned}
u^*(x,y) := \sup_{\theta\in\mathscr{T}} \Big\{ & \mathbb{E}_{\mathbb{Q}}\left[\int_0^{\tau\wedge\theta} e^{-rs} f(X(s),Y(s))\,ds\right] \\
&+ \mathbb{E}_{\mathbb{Q}}\left[e^{-r\theta}\psi(X(\theta),Y(\theta))1_{\{\theta<\tau\}}\right] \\
&+ \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}g(X(\tau),Y(\tau))1_{\{\tau\leq\theta\}}\right]\Big\}, \quad (x,y)\in\bar{\mathscr{O}},
\end{aligned}
$$

where $r > 0$, τ is the exit time from $\mathscr{O}$ of the process $(X(s), Y(s))_{s\geq 0}$ starting at $(x,y)\in\mathscr{O}$, and $\mathscr{T}$ is the set of $\mathbb{F}$-stopping times with values in $[0,\infty)$.

Probabilistic solutions to evolutionary var'l equalities I

Problem (Evolutionary variational equality)

Let $\mathscr{O}$ be as in Problem 4.1 and let $0 < T < \infty$ and $Q := [0, T) \times \mathscr{O}$. Let $f \in C(Q)$ obey

$$|f(t,x,y)| \leq C_1(1+y), \quad (t,x,y) \in Q,$$

let $g \in C_b((0,T) \times \partial\mathscr{O})$, and let $h \in C(\mathscr{O})$ obey

$$|h(x,y)| \leq C_2(1+e^{C_3 x}), \quad (x,y) \in \mathscr{O},$$

Find $u \in C^{1,2}(Q) \cap C(\bar{Q})$ such that

$$\begin{gathered} u' + Au = f \text{ on } Q, \\ u = g \text{ on } (0,T) \times \partial_\beta\mathscr{O}, \quad u(T,\cdot) = h \quad \text{on } \mathscr{O}. \end{gathered}$$

Probabilistic solutions to evolutionary var'l equalities II

Theorem (Uniqueness of solutions to Problem 4.5)

Let u be a solution to Problem 4.5. Then $u = u^*$ on $\bar{Q}$, with

$$u^*(t,x,y) := \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\tau} e^{-rs} f(X(s),Y(s))\,ds\right] + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)} h(X(T),Y(T)) 1_{\{\tau=T\}}\right] + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)} g(\tau,X(\tau),Y(\tau)) 1_{\{\tau<T\}}\right],$$

where $(t,x,y) \in Q$, $r > 0$, τ is the exit time from $\mathscr{O}$ of $(X(s),Y(s))_{s\geq t}$ starting at $(t,x,y) \in Q$, if such a time exists and $\tau = T$ otherwise.

Probabilistic solutions to evolutionary var'l inequalities I

Problem (Evolutionary variational inequality)

Let $\mathscr{O}$, T, Q, f, g, h be as in Problem 4.5, and let ψ obey

$$|\psi(t,x,y)| \leq C_4(1+e^{C_5 x}), \quad (t,x,y) \in Q,$$
$$\psi \leq g \text{ on } (0,T)\times\partial_\beta\mathscr{O} \quad \text{and} \quad \psi(T,\cdot) \leq h \text{ on } \mathscr{O}.$$

Find $u \in C^{1,2}(Q)\cap C(\bar{Q})$ such that

$$\begin{aligned} u' + Au &\geq f && \text{on } Q,\\ u &\geq \psi && \text{on } Q,\\ (u'+Au-f)(u-\psi) &= 0 && \text{on } Q,\\ u &= g && \text{on } (0,T)\times\partial_\beta\mathscr{O},\\ u(T,\cdot) &= h && \text{on } \mathscr{O}. \end{aligned}$$

Probabilistic solutions to evolutionary var'l inequalities II

Theorem (Uniqueness of solutions to Problem 4.7)

Let u be a solution to Problem 4.7. Then $u = u^*$ on $\bar{Q}$, with

$$
\begin{aligned}
u^*(t,x,y) := \sup_{\theta\in\mathscr{T}_{t,T}} \Bigg\{ & \mathbb{E}_{\mathbb{Q}}\left[\int_t^{\tau\wedge\theta} e^{-rs} f(s,X(s),Y(s))\,ds\right] \\
& + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\theta-t)}\psi(\theta,X(\theta),Y(\theta))1_{\{\theta<\tau\wedge T\}}\right] \\
& + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}h(X(T),Y(T))1_{\{T=\tau\wedge\theta\}}\right] \\
& + \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)}g(\tau,X(\tau),Y(\tau))1_{\{\tau\le\theta,\tau<T\}}\right]\Bigg\},
\end{aligned}
$$

where τ is exit time from $\mathscr{O}$ of $(X(s),Y(s))_{s\ge t}$ starting at $(t,x,y)\in Q$, and $\mathscr{T}_{t,T}$ is set of $\mathbb{F}$-stopping times valued in $[t,T]$.

Applications to finance

- In applications to option pricing, we need only consider solutions to the evolutionary variational inequality with $f = 0$, while $\psi(x, y) = (K - e^x)^+$ or $(e^x - K)^+$ and $h(x, y) = \psi(x, y)$.
- We denote $U(t, S, y) = u(t, x, y)$ and $\Psi(t, S, y) = \psi(t, x, y)$, where $S = e^x$.

The results on the next few slides provide a small sample of what may be proved by adapting arguments of Broadie & Detemple (1997), Jaillet, Lamberton, and Lapeyre (1990), Laurence and Salsa (2009), Touzi (1999), and Villeneuve (1999).

Properties of the solution

Lemma

Let $U(t, S, y)$ be as above. Then

1. $U(t, S, y)$ *is a non-increasing function of* $t \in [0, T]$.
2. *If* $\Psi(S)$ *is a convex function of* $S \in (0, \infty)$, *then* $U(t, S, y)$ *is a convex function of* $S \in (0, \infty)$, $\forall (t, y) \in [0, T] \times (0, \infty)$.
3. *If* $\Psi(S)$ *is a non-increasing (non-decreasing) function of* $S \in (0, \infty)$, *then* $U(t, S, y)$ *is a non-increasing (non-decreasing) function of* $S \in (0, \infty)$, $\forall (t, y) \in [0, T] \times (0, \infty)$.

Properties of the derivative

Lemma

Suppose $\Psi(S)$, $S \in (0, \infty)$, *obeys*

$$m(S_2 - S_1) \leq \Psi(S_2) - \Psi_1(S_1) \leq M(S_2 - S_1), \quad 0 < S_1 < S_2 < \infty,$$

for given $-\infty < m \leq M < \infty$. *Then, for each* $(t, y) \in [0, T) \times (0, \infty)$, $U(t, S, y)$ *is a differentiable function of* $S \in (0, \infty)$ *and*

$$m \leq \frac{\partial U}{\partial S} \leq M, \quad \forall (t, S, y) \in [0, T) \times (0, \infty) \times (0, \infty).$$

Continuation and exericise regions

Definition

Given a solution $U(t,S,y)$ to the evolutionary variational inequality for an obstacle function $\Psi(t,S,y)$, the continuation and exericise regions are defined by

$$\mathscr{C}(U) := \{(t,S,y) \in Q : U(t,S,y) > \Psi(t,S,y)\},$$
$$\mathscr{E}(U) := \{(t,S,y) \in Q : U(t,S,y) = \Psi(t,S,y)\},$$

and similarly for $\mathscr{C}(u)$ and $\mathscr{E}(u)$, given $u(t,x,y)$ and $\psi(t,x,y)$.

Characterization of the free boundary

The results of Touzi (1999) may be adapted to show

Proposition

If $\Psi(t,S,y)=(K-S)^+$*, there is a* $S^*:[0,T)\times(0,\infty)\to[0,K]$ *such that*

$$\mathscr{C}(U)=\{(t,S,y)\in[0,T)\times(0,\infty)\times(0,\infty): S>S^*(t,y)\}.$$

Lemma

If $\Psi(t,S,y)=(K-S)^+$*, then* $S^*:[0,T)\times(0,\infty)\to[0,K]$ *is decreasing with respect to* $y\in(0,\infty)$.

Properties of the free boundary

We expect, by work in progress, that

- $S^*(t, y)$ is a continuous function of $t \in [0, T)$, $\forall y \in (0, \infty)$.
- If $\Psi(S) = (K - S)^+$ (respectively, $(S - K)^+$), then $S^*(t, y)$ is a non-decreasing (respectively, non-increasing) function of $t \in [0, T)$, $\forall y \in (0, \infty)$.
- If $s^*(t, y) = \log S^*(t, y)$, then $s^*(t, \cdot)$ is Lipschitz, uniformly with respect to $t \in [0, T)$.
- $S^* : [0, T) \times (0, \infty) \to [0, K]$ is differentiable with respect to $y \in (0, \infty)$.

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Mathematical Finance and Partial Differential Equations 2010

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11 invited speakers and up to 6 contributed talks

THANK YOU!