On conditional McKean Langevin processes with no-permeability boundary condition

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Modeling turbulent flows :

The Reynolds averages (or ensemble averages) are expectations:

$$\langle \mathscr{U} \rangle(t,x) := \int_{\Omega} \mathscr{U}(t,x,\omega) d\mathbb{P}(\omega).$$

The corresponding Reynolds decomposition of the velocity is

$$\mathscr{U}(t,x,\omega) = \langle \mathscr{U} \rangle(t,x) + \mathbf{u}(t,x,\omega),$$
$$\mathscr{P}(t,x,\omega) = \langle \mathscr{P} \rangle(t,x) + \mathbf{p}(t,x,\omega)$$

The random field $\mathbf{u}(t,x,\omega)$ is the turbulent part of the velocity.

Incompressible Navier Stokes equation in $\mathbb{R}^3$, for the velocity field $(\mathscr{U}^{(1)}, \mathscr{U}^{(2)}, \mathscr{U}^{(3)})$ and the pressure $\mathscr{P}$, with constant mass density ρ

$$\partial_t \mathscr{U} + (\mathscr{U} \cdot \nabla)\mathscr{U} = \nu \Delta \mathscr{U} - \frac{1}{\rho} \nabla \mathscr{P}, \quad t > 0,\ x \in \mathbb{R}^3,$$
$$\nabla \cdot \mathscr{U} = 0, \quad t \geq 0,\ x \in \mathbb{R}^3,$$
$$\mathscr{U}(0,x) = \mathscr{U}_0(x),\ x \in \mathbb{R}^3.$$

The Reynolds averaged equation for the mean velocity

Assuming Reynolds decomposition, we obtain the unclosed equation with constant mass density ρ

$$\partial_t \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \langle \mathscr{U}^{(j)} \rangle \partial_{x_j} \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \partial_{x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \nu \Delta \langle \mathscr{U}^{(i)} \rangle - \frac{1}{\rho} \partial_{x_i} \langle \mathscr{P} \rangle,$$

$$\nabla \cdot \langle \mathscr{U} \rangle = 0, \; t \geq 0, \; x \in \mathbb{R}^3,$$

$$\langle \mathscr{U} \rangle (0, x) = \langle \mathscr{U}_0 \rangle (x), \; x \in \mathbb{R}^3,$$

where $\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle - \langle \mathscr{U}^{(i)} \rangle \langle \mathscr{U}^{(j)} \rangle$.

Direct modeling of the Reynolds stress by a turbulent viscosity model:

$$\text{kinetic turbulent energy } k(t, x) := \sum_{i=1}^{3} \frac{1}{2} \langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle (t, x)$$

and

$$\text{pseudo-dissipation } \varepsilon(t, x) := \nu \sum_{i=1}^{3} \sum_{j=1}^{3} \langle \partial_{x_j} \mathbf{u}^{(i)} \partial_{x_j} \mathbf{u}^{(i)} \rangle (t, x).$$

The equation for the Reynolds stress $(\langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle, i, j)$

$$
\partial_t \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle + \left(\langle \mathscr{U}\rangle \cdot \nabla_x \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle\right) + \sum_{k=1}^{3} \partial_{x_k} \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\mathbf{u}^{(k)}\rangle
$$
$$
= -\frac{1}{\rho}\langle \mathbf{u}^{(j)}\partial_{x_i}\mathbf{p} + \mathbf{u}^{(i)}\partial_{x_j}\mathbf{p}\rangle + \nu \sum_{k=1}^{3} \partial^2_{x_k} \langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle
$$
$$
+ \nu \sum_{k=1}^{3} \langle \partial_{x_k}\mathbf{u}^{(i)}\partial_{x_k}\mathbf{u}^{(j)}\rangle - \sum_{k=1}^{3} \left(\langle \mathbf{u}^{(i)}\mathbf{u}^{(k)}\rangle \partial_{x_k}\langle \mathscr{U}^{(j)}\rangle + \langle \mathbf{u}^{(j)}\mathbf{u}^{(k)}\rangle \partial_{x_k}\langle \mathscr{U}^{(i)}\rangle\right).
$$

Higher order closure : model equation for the Reynolds stress.

An alternative approach to compute the Reynolds stress [Stephen B. Pope]

Let $f_E(t,x;V)$ be the Probability Density Function (PDF) of the random field $\mathscr{U}(t,x)$, then

$$\langle \mathscr{U}^{(i)} \rangle (t,x) = \int_{\mathbb{R}^3} V^{(i)} f_E(t,x;V) dV,$$
$$\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t,x) = \int_{\mathbb{R}^3} V^{(i)} V^{(j)} f_E(t,x;V) dV.$$

The closure problem is reported on the PDE satisfied by the probability density function f_E.

In a series of papers (see e.g. Pope 85), Stephen B. Pope propose to model the PDF f_E with a Lagrangian description of the flow.

Fluid particle model family

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the fluid particle state vector $(\mathrm{X}_t, \mathrm{U}_t, \psi_t)$ satisfying

$$
\begin{aligned}
d\mathrm{X}_t &= \mathrm{U}_t dt, \\
d\mathrm{U}_t &= \left[-\frac{1}{\rho} \nabla_x \langle \mathscr{P} \rangle (t, \mathrm{X}_t) + \nu \triangle_x \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right] dt \\
&\quad - G(t, \mathrm{X}_t) \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right) dt + \sqrt{C(t, \mathrm{X}_t) \varepsilon (t, \mathrm{X}_t)} dW_t, \\
d\psi_t &= D_1(t, \mathrm{X}_t, \psi_t) dt + D_2(t, \mathrm{X}_t, \psi_t) d\widetilde{W}_t.
\end{aligned}
$$

$\left(W, \widetilde{W}\right)$ is a $4D$-Brownian motion.

One needs to

- compute de Eulerian fields $\langle \mathscr{U}^{(i)} \rangle (t, x)$, $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t, x)$.
- determine ε, C, G, D_1, D_2 by the RANS closure.

Compute the Reynolds averages $\langle \mathscr{U}^{(i)} \rangle$ *and* $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle$

Let $f_L(t; x, V, \psi)$ be the probability density function of $(\mathrm{X}_t, \mathrm{U}_t, \psi_t)$.

Eulerian PDF versus Lagrangian PDF:

(Case of incompressible flow with constant mass density)

$$f_E(t, x; V, \phi) = \frac{f_L(t; x, V, \phi)}{\displaystyle\int_{\mathbb{R}^4} f_L(t; x, V, \psi) dV d\psi}$$

For any bounded measurable function $F(v)$,

$$\langle F(\mathscr{U}) \rangle (t, x) = \mathbb{E}\left(F(\mathrm{U}_t) / \mathrm{X}_t = x \right).$$

In particular,

$$\left\langle \mathscr{U}^{(i)} \right\rangle (t, x) = \int_{\mathbb{R}^4} V^{(i)} \frac{f_L(t; x, V, \phi)}{\displaystyle\int_{\mathbb{R}^4} f_L(t; x, U, \psi) dU d\psi} dV d\phi = \mathbb{E}\left(\mathrm{U}_t^{(i)} / \mathrm{X}_t = x \right).$$

The Simplified model (Pope 94)

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho}\frac{\partial\langle\mathscr{P}\rangle}{\partial x_i}(t,\mathrm{X}_t) \\ & \quad - \left(\frac{1}{2}+\frac{3}{4}C_0\right)\frac{\varepsilon(t,\mathrm{X}_t)}{k(t,\mathrm{X}_t)}\left(\mathrm{U}_t^{(i)} - \langle\mathscr{U}^{(i)}\rangle(t,\mathrm{X}_t)\right)\Big] dt \\ & + \sqrt{C_0\varepsilon(t,\mathrm{X}_t)}dW_t^{(i)}, \quad i \in \{1,2,3\} \end{cases}$$

+ boundary conditions + wall boundary functions.

$\varepsilon(t,x)$ and $k(t,x)$ are supposed to be known.

$\langle\mathscr{P}\rangle(t,x)$ must be recovered by the Poisson equation

$$\nabla^2\langle\mathscr{P}\rangle = -\frac{\partial^2\langle\mathscr{U}^{(i)}\mathscr{U}^{(j)}\rangle}{\partial x_i \partial x_j}$$

which guarantees that the averaged Eulerian velocity is divergence free.

The Stochastic Downscaling Method in $\mathcal{D} \subset \mathbb{R}^3$

$$
\begin{cases}
dX_t = & U_t dt, \\
dU_t^{(i)} = & \Big[-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, X_t) \\
& \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, X_t)}{k(t, X_t)} \left(U_t^{(i)} - \langle \mathscr{U}^{(i)} \rangle (t, X_t) \right) \Big] dt \\
& + \sqrt{C_0 \varepsilon(t, X_t)} dW_t^{(i)}, \quad i \in \{1, 2, 3\} \\
& \\
& + \text{ boundary conditions on } \partial\mathcal{D}.
\end{cases}
$$

- $k(t,x) = \frac{1}{2}(\langle \mathscr{U}^{(i)} \mathscr{U}^{(i)} \rangle - \langle \mathscr{U}^{(i)} \rangle^2$ is computed inside the model.

- $\langle \mathscr{P} \rangle (t,x)$ must be recovered by the Poisson equation

$$
\nabla^2 \langle \mathscr{P} \rangle = - \frac{\partial^2 \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle}{\partial x_i \partial x_j}
$$

which guarantees that the averaged Eulerian velocity is divergence free.

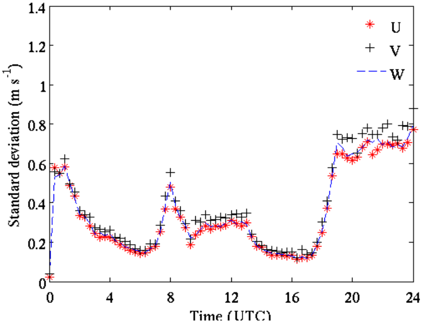
The turbulent-kinetic-energy model

SDM is running with a common closure for atmospheric flows.
Objetive : refine the computation of wind prediction from a classical meteorological solver.

- The mixing length $\ell_m = \ell_m(z)$
- the turbulent viscosity $\nu_T = \frac{C_k}{\ell_m} k^{1/2}$
- A model for the dissipation rate: $\varepsilon(t,x,y,z) = \frac{C_\varepsilon}{\ell_m(z)} k^{3/2}(t,x,y,z)$

Figure: Square root of $\langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle$, for $i = 1, 2, 3$ in one cell of $\mathcal{D}$.

Initial condition should satisfy the guessed physical behaviour. ($\Rightarrow k$)



The Guidance with an external velocity field

The Downscaling method

Let $\mathcal{D}$ be an open set of $\mathbb{R}^3$, and a velocity V_{ext} given at $\partial\mathcal{D}$:

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t = & \Big[-\frac{1}{\rho}\nabla\langle\mathscr{P}\rangle(t,\mathrm{X}_t) \\ & \quad - \left(\frac{1}{2}+\frac{3}{4}C_0\right)\frac{\varepsilon(t,\mathrm{X}_t)}{k(t,\mathrm{X}_t)}\left(\mathrm{U}_t - \langle\mathscr{U}\rangle(t,\mathrm{X}_t)\right)\Big] dt \\ & + \sqrt{C_0\varepsilon(t,\mathrm{X}_t)}dW_t \\ & + \sum_{0<s\leq t} 2\left(V_{\text{ext}}(s,\mathrm{X}_s) - \mathrm{U}_{s^-}\right)\mathbb{1}_{\{\mathrm{X}_s\in\partial\mathcal{D}\}}. \end{cases}$$

The jump term should ensure that

$$\langle\mathscr{U}\rangle(t,x) = V_{\text{ext}}(t,x), \forall x\in\partial\mathcal{D}.$$

The Guidance with an external velocity field

Boundary condition

For all x at the boundary $\partial\mathcal{D}$,

$$\langle \mathscr{U} \rangle(t,x) = V_{\text{ext}}(t,x).$$

$$\frac{\int_{\mathbb{R}^d} v\rho(t,x,v)dv}{\int_{\mathbb{R}^d} \rho(t,x,v)dv} = V_{\text{ext}}(t,x).$$

If $\int_{\mathbb{R}^d} \rho(t,x,v)dv > 0$

$$\begin{aligned} & \int_{\mathbb{R}^d} v\rho(t,x,v)dv = \int_{\mathbb{R}^d} V_{\text{ext}}(t,x)\rho(t,x,v)dv \\ \Leftrightarrow \quad & \int_{\mathbb{R}^d} v\rho(t,x,v)dv = \int_{\mathbb{R}^d} v\rho(t,x,v+2(V_{\text{ext}}(t,x)-v))dv \\ \Uparrow \quad & \end{aligned}$$

If $\rho(t,x,v) = \rho(t,x,v+2(V_{\text{ext}}(t,x)-v)),\ \forall v \in \mathbb{R}^d.$

Mathematical study of a simplified Langevin model

$2d$ dimensional SDE in the phase space (position, velocity) :

$$\begin{cases} d\mathrm{X}_t = \mathrm{U}_t dt, \\ d\mathrm{U}_t = \mathbb{E}\left[b(u, \mathrm{U}_t)/\mathrm{X}_t\right]\big|_{u=\mathrm{U}_t} dt + dW_t, \ t \in [0, T]. \end{cases}$$

Nonlinear drift term in the sense of McKean.

Related works:

Sznitman (86) : Propagation of chaos for the Burgers Equation :

$$X_t = X_0 + W_t + 2\int_0^t u(s, X_s) ds$$

$u(t,x)dx$ is the law of X_t.

Dermoune (03) : Conditional propag. of chaos for pressurless gas Eq.

$$X_t = X_0 + W_t + \int_0^t \mathbb{E}[v(X_0)/X_s] ds.$$

Here : local interaction in the d first variables $(x_1, \ldots, x_d)$. Hypoelliptic Fokker-Plank equation. We need a Propagation of chaos result.

Spatially Confined Langevin model in $\mathcal{D} \subset \mathbb{R}^d$ with a mean no-permeability condition

Homogeneous Dirichlet condition for the impact problem with stochastic forcing :

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t \mathbb{E}[b(u, \mathrm{U}_s)/\mathrm{X}_t]\big|_{u=\mathrm{U}_s} ds + W_t \\ & - \sum_{0<s\leq t} 2\left(\mathrm{U}_{s-} \cdot n_{\mathcal{D}}(\mathrm{X}_s)\right) n_{\mathcal{D}}(\mathrm{X}_s) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}} \end{cases}$$

must satisfy the averaged no-permeability condition : $\forall x \in \partial\mathcal{D}$,

$$(\langle \mathscr{U}(t,x) \rangle \cdot n_{\mathcal{D}}(x)) = \mathbb{E}\left[(\mathrm{U}_t \cdot n_{\mathcal{D}}(\mathrm{X}_t))/\mathrm{X}_t = x\right] = 0.$$

Spatially Confined Langevin model in $\mathcal{D} \subset \mathbb{R}^d$

$(\mathrm{X}, \mathrm{U}) \in C([0,T]; \overline{\mathcal{D}}) \times \mathbb{D}([0,T]; \mathbb{R}^d)$

$$\begin{cases} \mathrm{X}_t = \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = \mathrm{U}_0 + \int_0^t \mathbb{E}[b(u, \mathrm{U}_s)/\mathrm{X}_t]\big|_{u=\mathrm{U}_s} ds + W_t + K_t, \\ K_t = -\sum_{0<s\leq t} 2 \left(\mathrm{U}_{s^-} \cdot n_{\mathcal{D}}(\mathrm{X}_s)\right) n_{\mathcal{D}}(\mathrm{X}_s) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}}, \end{cases} \tag{1}$$

We are interested in the solution of (1) such that the sequence of hitting times

$$\begin{cases} \tau_0 = 0, \\ \tau_n = \inf\{t > \tau_{n-1} \text{ s.t. } \mathrm{X}_t \in \partial\mathcal{D}\}, \text{ for } n \geq 1, \end{cases}$$

is well-defined and grows to infinity.

Spatially Confined Langevin model in $\mathcal{D} \subset \mathbb{R}^d$

$(X, \mathrm{U}, K) \in C([0,T]; \overline{\mathcal{D}}) \times \mathbb{D}([0,T]; \mathbb{R}^d) \times \mathbb{D}([0,T]; \mathbb{R}^d)$ s.t.

$$\begin{cases} \mathrm{X}_t = \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = \mathrm{U}_0 + \int_0^t B\left[\mathrm{X}_s, \mathrm{U}_s; \rho_s\right] ds + W_t + K_t, \\ K_t = - \sum_{0<s\leq t} 2\left(\mathrm{U}_{s^-} \cdot n_{\mathcal{D}}(\mathrm{X}_s)\right) n_{\mathcal{D}}(\mathrm{X}_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}, \\ \rho_t \text{ is the Lebesgue density of } (X_t, \mathrm{U}_t),\ \forall t \in (0,T], \end{cases}$$

where the drift coefficient $B : \mathcal{D} \times \mathbb{R}^d \times L^1(\mathcal{D} \times \mathbb{R}^d) \to \mathbb{R}^d$ is defined by

$$B\left[x, u; \gamma\right] = \begin{cases} \dfrac{\int_{\mathbb{R}^d} b(v,u)\gamma(x,v)\,dv}{\int_{\mathbb{R}^d} \gamma(x,v)\,dv} & \text{if } \int_{\mathbb{R}^d} \gamma(x,v)\,dv \neq 0, \\ 0 \text{ elsewhere,} \end{cases}$$

Formally, $B[x, u; \rho_t] = \mathbb{E}\left[b(\mathrm{U}_t, u) / \mathrm{X}_t = x\right]$.

Confined Langevin model

Well-posedness in the hyperplane $\mathcal{D} = \mathbb{R}^{d-1} \times \mathbb{R}^+$:

Our hypotheses (*H*):

(i) The initial measure μ_0 has its support in $\mathcal{D} \times \mathbb{R}^d$ and

$$\int_{\mathcal{D}\times\mathbb{R}^d} \left(|x| + |u|^2\right) \mu_0(dx, du) < +\infty.$$

(ii) $b : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is uniformly bounded and continuous.

Impact problem in the deterministic case

writes as a multivalued ODE

$$\ddot{u} \in b(\cdot, u, \dot{u}) - \partial\phi(u)$$

where $\partial\phi$ is the subdifferential of a convex function
Uniqueness has been proven only when b is a harmonic function (see Schatzman 98, Ballard 2001).

Confined Langevin model in $\mathcal{D} = \mathbb{R}^{d-1} \times \mathbb{R}^+$

$\mathcal{E} := C([0,T];\overline{\mathcal{D}}) \times \mathbb{D}([0,T];\mathbb{R}^d) \times \mathbb{D}([0,T];\mathbb{R}^d)$, equiped with the Skorokhod topology, $(x_t, u_t, k_t; t \in [0,T])$ its canonical process.

Theorem

Assume (H). Then there exists a unique solution $\mathbb{P} \in \mathcal{M}(\mathcal{E})$ *to the following martingale problem (MP) :*

(i) $\mathbb{P} \circ (x_0, u_0, k_0)^{-1} = \mu_0 \otimes \delta_0$.

(ii) $\forall t \in (0,T]$, $\mathbb{P} \circ (x_t, u_t)^{-1}$ *admits a positive Lebesgue density* ρ_t.

(iii) *For all* $f \in C_b^2(\mathbb{R}^{2d})$, *the process*

$$f(x_t, u_t - k_t) - f(x_0, u_0) - \int_0^t (u_s \cdot \nabla_x f(x_s, u_s - k_s))\, ds$$
$$- \int_0^t \left[(B[x_s, u_s; \rho_s] \cdot \nabla_u f(x_s, u_s - k_s)) + \frac{1}{2} \triangle_u f(x_s, u_s - k_s) \right] ds$$

is a continuous $\mathbb{P}$*-martingale w.r.t.* $(\mathcal{B}_t; t \in [0,T])$.

(iv) $\mathbb{P}$*-a.s., the set* $\{t \in [0,T] \text{ s.t. } x_t = 0\}$ *is at most countable, and*

$$k_t = -2 \sum_{0 < s \le t} u_{s^-}^{(d)} \mathbb{1}_{\left\{x_s^{(d)} = 0\right\}}, \ \forall\, t \in [0,T].$$

On the mean no-permeability condition

Proposition

(a) The densities (ρ_t) related to (MP) are Hölder-continuous : $\exists\ \beta$, α, α_1 positive, for a.e. $0 < t_0 < t \leq T,\ u \in \mathbb{R}^d$,

$$|\rho_t(x,u) - \rho_t(x_0,u)| \leq C\left(t_0^{-\alpha}(1 \vee (t-t_0)^{\alpha_1}\right)|x-x_0|^\beta,$$

(b) For a.e. $(t,x,u) \in \Sigma_T$, $\gamma(\rho)(t,x,u) := \lim_{\delta\to 0^+} \rho_t((x',\delta),u)$ satisfies: $\forall f \in \mathcal{B}_b((0,T)\times\mathbb{R}^{2d})$,

$$\mathbb{E}_{\mathbb{P}}\left[\sum_{n\in\mathbb{N}}\Big(f(\tau_n, x_{\tau_n}, u_{\tau_n}) - f(\tau_n, x_{\tau_n}, u_{\tau_n^-})\Big)\,\mathbb{1}_{\{\tau_n \leq t\}}\right]$$
$$= -\int_{\Sigma_t}(u\cdot n_{\mathcal{D}}(x))\,\gamma(\rho)(s,(x',0),u)f(s,(x',0),u)\,ds\,dx'du.$$

From (b), $\gamma(\rho)$ satisfies the specular boundary condition:
$\gamma(\rho)(t,x,u) = \gamma(\rho)(t,x,u - 2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)),\ dt\otimes d\sigma_{\mathcal{D}}\otimes du$-a.e. on Σ_T

On the mean no-permeability condition

Corollary

- $\displaystyle\int_{\mathbb{R}^d} |(v \cdot n_{\mathcal{D}}(x))| \gamma(\rho)(t,x,v)\, dv < +\infty,$
- $\displaystyle\int_{\mathbb{R}^d} \gamma(\rho)(t,x,v)\, dv > 0, \qquad dt \otimes d\sigma_{\mathcal{D}}\text{-a.e. on } (0,T) \times \partial\mathcal{D}.$

Then

$$\mathbb{E}_{\mathbb{P}}\left[(\mathrm{U}_t \cdot n_{\mathcal{D}}(x))/\mathrm{X}_t = x\right] = \frac{\displaystyle\int_{\mathbb{R}^d} (u \cdot n_{\mathcal{D}}(x)) \gamma(\rho)(t,x,u)\, du}{\displaystyle\int_{\mathbb{R}^d} \gamma(\rho)(t,x,u)\, du}$$

and the mean no-permeability condition is fulfilled.

The confined Brownian motion primitive in the half line

Starting from $(\mathrm{X}_0, \mathrm{U}_0)$ with $\mathrm{X}_0 > 0$, and a (B_t) Brownian motion in $\mathbb{R}$,

$$\mathscr{Y}_t = \mathrm{X}_0 + \int_0^t \mathscr{V}_s ds, \quad \mathscr{V}_t = \mathrm{U}_0 + B_t.$$

Set $\mathrm{X}_t = |\mathscr{Y}_t|$ and

$$\mathrm{U}_t = \mathscr{V}_t \mathscr{S}_{t^+}, \text{ with } \mathscr{S}_t := sign(\mathscr{Y}_t).$$

Lemma

If ρ_0 has its support in $\mathbb{R} \times (0, +\infty) \times \mathbb{R}$, then $\mathscr{S}_t$ jumps a countable number of times, and U_t solves

$$\mathrm{U}_t = \mathrm{U}_0 + W_t - 2 \sum_{0<s\leq t} \mathrm{U}_{s-} \mathbb{1}_{\{\mathrm{X}_s = 0\}} \; a.s.$$

where W_t is a Brownian motion.

(Lachal 97: Passage time of the Brownian motion primitive at 0)

The Brownian motion primitive in $\mathbb{R}^d$

- $g_d(t; z, \nu; \zeta, \mu)$
$$= \left(\frac{\sqrt{3}}{\pi t^2}\right)^d \exp\left(-\frac{6|\zeta - z - t\nu|^2}{t^3} + \frac{6((\zeta - z - t\nu)\cdot(\mu - \nu))}{t^2} - \frac{2|\mu - \nu|^2}{t}\right)$$

- $$\sup_{(y,v)\in\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |\nabla_v g_d(t; y, v; x, u)| \, dx\, du \leq \frac{C}{\sqrt{t-s}}.$$

Lemma [Manfredini Polidoro 98]

For $p > 4d + 2$, let $h \in L^p(\mathbb{R}^{2d})$ and $H \in L^p((0, T) \times \mathbb{R}^{2d})$. Then, for all $0 < t_0 < t \leq T$, it holds that

$$\int_{\mathbb{R}^{2d}} |g_d(t - t_0; y, v; x, u) - g_d(t - t_0; y, v; x_0, u)| \, |h(y, v)| \, dy\, dv \leq c|x - x_0|^{\frac{1}{3} - \frac{4d+1}{3p}} \|h\|_{L^p(\mathbb{R}^2}$$

$$\int_{(t_0,t)\times\mathbb{R}^{2d}} |\nabla_v g_d(t - s; y, v; x, u) - \nabla_v g_d(t - s; y, v; x_0, u)| \, |H(s, y, v)| \, dy\, dv\, ds$$
$$\leq c|x - x_0|^{\frac{1}{3} - \frac{4d+2}{3p}} \|H\|_{L^p((t_0,t)\times\mathbb{R}^{2d})}.$$

Transition of the confined Brownian motion primitive ...

For any point $x \in \mathbb{R}^d$, we note $x = (x', x^{(d)})$, $x' \in \mathbb{R}^{d-1}$, $x^{(d)} \in \mathbb{R}$.

$$\Gamma(t; y, v; x, u) = g_{d-1}(t; y', v'; x', u') \Big(g_1(t; y^{(d)}, v^{(d)}; x^{(d)}, u^{(d)}) + g_1(t; y^{(d)}, v^{(d)}; -x^{(d)}, -u^{(d)})\Big)$$

Define

$$f \mapsto S_t(f)(y, v) = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t; y, v; x, u) f(x, u)\, dx\, du$$

$$\mu \mapsto S^*_t(\mu)(x, u) = \int_{\mathcal{D}\times\mathbb{R}^d} \Gamma(t; y, v; x, u)\mu(dy, dv)$$

$$\phi \mapsto S'_t(\phi)(x, u) = \int_{\mathcal{D}\times\mathbb{R}^d} (\nabla_v \Gamma(t; y, v; x, u) \cdot \phi(y, v))\, dy\, dv.$$

...and the McKean Nonlinear mild equation

The set of time-marginal densities $(\rho(t,\cdot,\cdot))$ related to (MP)

- is the unique solution of the mild equation in $L^1(\mathcal{D}\times\mathbb{R}^d)$:

$$\forall\ t\in(0,T]\,,\ \rho(t,x,v)=S_t^*(\mu_0)(x,v)+\int_0^t S'_{t-s}(\rho(s))(x,v)B[\cdot;\rho(s)])\,ds$$

- $x\mapsto\rho(t,x,v)$ is Hölder-continuous, and

Lemma

$\rho(t,x,v)$ is the unique weak solution of the following Vlasov-Fokker-Planck Equation with specular boundary condition :

$$\begin{cases} \dfrac{\partial\rho}{\partial t}+v\cdot\nabla_x\rho+\left(\left[\dfrac{\int_{\mathbb{R}^2} b(v,u)\rho(t,x,u)du}{\int_{\mathbb{R}^2}\rho(t,x,u)du}\right]\cdot\nabla_v\rho\right)=\dfrac{1}{2}\Delta_v\rho, \\ \hfill (t,x,v)\in(0,T)\times\mathcal{D}\times\mathbb{R}^d, \\ \rho(0,x,v)=\rho_0(x,v)\ \text{given}, (x,v)\in\mathcal{D}\times\mathbb{R}^d, \\ \rho(t,x,v)=\rho(t,x,v-2(v\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \\ \hfill (t,v)\in(0,T)\times\mathbb{R}^d, x\in\partial\mathcal{D}. \end{cases}$$

A smoothed system in the space variables

$\forall$ non–negative γ in $L^1(\mathbb{R}^{2d})$,

$$B_\varepsilon[x,u;\gamma] = \frac{\int_{\mathbb{R}^{2d}} b(v,u)\phi_\varepsilon(x-y)\gamma(y,v)\,dy\,dv}{\int_{\mathbb{R}^{2d}} \phi_\varepsilon(x-y)\gamma(y,v)\,dy\,dv + \varepsilon},$$

$\phi_\epsilon(x) := \epsilon^{-d}\phi(\frac{x}{\epsilon})$ for some $\phi \in C^1_c(\mathcal{D})$.
We construct a solution to (MP) by means of a smoothed and confined interacting particle system :

$$\begin{cases} \mathrm{X}^{i,\epsilon,N}_t = \mathrm{X}^i_0 + \int_0^t \mathrm{U}^{i,\epsilon,N}_s ds, \\ \mathrm{U}^{i,\epsilon,N}_t = \mathrm{U}^i_0 + \int_0^t B_\varepsilon[\mathrm{X}^{i,\epsilon,N}_s, \mathrm{U}^{j,\epsilon,N}_s; \bar{\mu}^{\epsilon,N}_s]ds + W^i_t + K^{i,\epsilon,N}_t, \ i = 1,\dots,N, \\ K^{i,\epsilon,N}_t = -2\sum_{0<s\le t} \left(\mathrm{U}^{i,\epsilon,N}_{s^-}\cdot n_{\mathcal{D}}(\mathrm{X}^{i,\epsilon,N}_s)\right) n_{\mathcal{D}}(\mathrm{X}^{i,\epsilon,N}_s)\, \mathbb{1}_{\left\{\mathrm{X}^{i,\epsilon,N}_s \in \partial\mathcal{D}\right\}}, \end{cases}$$

$(\mathrm{X}^i_0, \mathrm{U}^i_0, W^i)_{i\ge 1}$ are independent copies of $(\mathrm{X}_0, \mathrm{U}_0, W)$.
$\bar{\mu}^{\epsilon,N} := \frac{1}{N}\sum_{i=1}^N \delta_{\left\{\mathrm{X}^{i,\epsilon,N}, \mathrm{U}^{i,\epsilon,N}\right\}}$

Convergences

In N : The sequence $\{\pi^N = \mathcal{L}aw\left(\frac{1}{N}\sum_{i=1}^N \delta_{\{\mathrm{X}^{i,\varepsilon,N},\mathrm{U}^{i,\varepsilon,N},K^{i,\varepsilon,N}\}}\right), N \in \mathbb{N}\}$ is tight on $\mathcal{P}(C([0,T];\overline{\mathcal{D}} \times \mathbb{D}([0,T];\mathbb{R}^d) \times \mathbb{D}([0,T];\mathbb{R}^d))$.

In ε : Convergence of the solution of the martingale problem (MP_ε) associated to the smoothed McKean SDE

$$\begin{cases} \mathrm{X}_t^\varepsilon = \mathrm{X}_0 + \int_0^t \mathrm{U}_s^\varepsilon ds, \\ \mathrm{U}_t^\varepsilon = \mathrm{U}_0 + \int_0^t B_\varepsilon\left[\mathrm{X}_s^\varepsilon, \mathrm{U}_s^\varepsilon; \rho_s\right] ds + W_t + K_t^\varepsilon, \\ K_t^\varepsilon = -\sum_{0<s\leq t} 2\left(\mathrm{U}_{s^-}^\varepsilon \cdot n_{\mathcal{D}}(\mathrm{X}_s^\varepsilon)\right) n_{\mathcal{D}}(\mathrm{X}_s)\, \mathbb{1}_{\{\mathrm{X}_s^\varepsilon \in \partial\mathcal{D}\}}, \\ \rho_t^\varepsilon \text{ is the density distribution of } (\mathrm{X}_t^\varepsilon, \mathrm{U}_t^\varepsilon), \text{ for all } t \in (0,T]. \end{cases}$$

to the solution of (MP).

And for other domains ?

Theorem

$\mathcal{D}$ a smooth bounded domain in $\mathbb{R}^d$.
$b : \mathbb{R}^{2d} \to \mathbb{R}^d$ bounded.
Weak existence (in $L^2\left((0,T)\times\mathcal{D}; H^1(\pi,\mathbb{R}^d)\right)$) and uniqueness of the Vlasov-Fokker-Planck Equation with specular boundary condition :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + v\cdot\nabla_x\rho + \left(\left[\dfrac{\int_{\mathbb{R}^d} b(v,u)\rho(t,x,u)du}{\int_{\mathbb{R}^d}\rho(t,x,u)du}\right]\cdot\nabla_v\rho\right) = \dfrac{1}{2}\Delta_v\rho, \\ \qquad\qquad\qquad\qquad (t,x,v)\in(0,T)\times\mathcal{D}\times\mathbb{R}^d, \\ \rho(0,x,v) = \rho_0(x,v),\ (x,v)\in\mathcal{D}\times\mathbb{R}^d, \\ \rho(t,x,v) = \rho(t,x,v-2(v\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \\ \qquad\qquad\qquad\qquad (t,v)\in(0,T)\times\mathbb{R}^d, x\in\partial\mathcal{D}. \end{cases}$$

Propagation of initial Maxwellian bounds for the sub- and super-solutions.

$\pi(u) = (1+|u|^2)^{\frac{3d}{2}}$

A weak notion of trace

$$\Sigma_t^{\pm} = \{(s,x,u) \in (0,T) \times \partial\mathcal{D} \times \mathbb{R}^d ; \pm(u \cdot n_{\mathcal{D}}(x)) > 0\}$$

Moreover

ρ admits traces $\gamma^{\pm}(\rho)$ on $\Sigma_T^{\pm}$ such that $\gamma^{\pm}(\rho) \in L^1(\pi, \Sigma_T^{\pm})$ and
For all $t \in (0,T)$, for all f in $C_c^{\infty}([0,T] \times \bar{\mathcal{D}} \times \mathbb{R}^d)$, the following Green formula holds

$$\int_{Q_t} \rho(-\partial_s f - u \cdot \nabla_x f) + f(B[;\rho] \cdot \nabla_u \rho) + \frac{\sigma^2}{2}(\nabla_u f \cdot \nabla_u \rho) ds\, dx\, du$$

$$= -\int_{\mathcal{D}\times\mathbb{R}^d} f(t,x,u)\rho(t,x,u)\, dx\, du + \int_{\mathcal{D}\times\mathbb{R}^d} f(0,x,u)\rho_0(x,u)\, dx\, du$$

$$-\int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x))\gamma^+(\rho)(s,x,u) f(t,x,u)\, ds\, d\sigma_{\mathcal{D}}(x) du$$

$$-\int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x))\gamma^+(\rho)(s,x,u - 2(u \cdot n_{\mathcal{D}}(x)) n_{\mathcal{D}}(x)) f(t,x,u)\, ds\, d\sigma_{\mathcal{D}}(x) du$$

Maxwellian bounds

Our hypotheses :

(i) b is Borel bounded.

(i) $\rho_0(x,\cdot) \in L^1(\mathbb{R}^d)$ and $\rho_0^2 \in L^1(\pi, \mathcal{D} \times \mathbb{R}^d)$.

(ii) There exists two functions $\underline{P}_0$, $\overline{P}_0$ from $\mathbb{R}^+$ to $\mathbb{R}^+$, with $\overline{P}_0^2 \in L^1(\pi, \mathcal{D} \times \mathbb{R}^d)$, s.t.

$$\underline{P}_0 \leq \rho_0(x,u) \leq \overline{P}_0, \text{ a.e on } \mathcal{D} \times \mathbb{R}^d.$$

Then

Maxwellian lower and upper bounds :

There exists a couple of Maxwellian $(\underline{P}, \overline{P})$ s.t.

$$\underline{P} \leq \rho \leq \overline{P}, \text{ a.e. on } Q_T$$
$$\underline{P} \leq \gamma^{\pm}(\rho) \leq \overline{P}, \text{ a.e on} \Sigma_T^{\pm}.$$

Construction of the process

$$\begin{cases} \mathrm{X}_t = \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = \mathrm{U}_0 + \int_0^t B\left[\mathrm{X}_s, \mathrm{U}_s; \rho_s\right] ds + W_t + K_t, \\ K_t = - \sum_{0<s\leq t} 2\left(\mathrm{U}_{s^-} \cdot n_{\mathcal{D}}(\mathrm{X}_s)\right) n_{\mathcal{D}}(\mathrm{X}_s) \mathbb{1}_{\{X_s \in \partial\mathcal{D}\}}, \\ \rho_t \text{ is the Lebesgue density of } (X_t, \mathrm{U}_t), \ \forall t \in (0, T], \end{cases}$$

We want to prove that

$$\mathbb{P}((\mathrm{X}_{\tau_1}, \mathrm{U}_{\tau_1}) \in \Sigma_0) = 0$$

$$\lim_{n \to +\infty} \mathbb{P}(\tau_n > t) = 1.$$

And for outward velocity guidance ?

When $\mathcal{D}$ is the half line:

For any smooth bounded F

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + F(\mathrm{X}_0) + W_t - \sum_{0<s\leq t} 2\mathrm{U}_{s-} \mathbb{1}_{\{\mathrm{X}_s=0\}} \end{cases}$$

becomes

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t (\mathrm{V}_s - F(\mathrm{X}_s)) ds, \\ \mathrm{V}_t = & \mathrm{U}_0 + B_t - \sum_{0<s\leq t} 2(\mathrm{V}_{s-} - F(0)) \mathbb{1}_{\{\mathrm{X}_s=0\}} \end{cases}$$

by setting $W_t = B_t - \int_0^t F'(Y_s) \mathrm{U}_s ds$ and $\mathrm{V}_t = \mathrm{U}_t + F(\mathrm{X}_t)$.

For the choice $F(x) = A \mathbb{1}_{\{x\leq 0\}}$, the free process is still well define.

Smoothing :

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t (\mathrm{V}_s - F * \phi_\epsilon(\mathrm{X}_s)) ds, \\ \mathrm{V}_t = & \mathrm{U}_0 + B_t - \sum_{0<s\leq t} 2(\mathrm{V}_{s-} - F * \phi_\epsilon(0)) \mathbb{1}_{\{\mathrm{X}_s=0\}} \end{cases}$$

Euler scheme for linear confined models, $\mathcal{D} = \mathbb{R}^+$

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t b(\mathrm{U}_s) ds + W_t - \sum_{0<s\leq t} 2\mathrm{U}_{s-} \mathbb{1}_{\{\mathrm{X}_s = 0\}}. \end{cases}$$

Euler scheme : $\Delta t > 0$ and $K \in \mathbb{N}$ s.t. $T = K\Delta t$; $t_k := k\Delta t$, $1 \leq k \leq K$, $(\bar{\mathrm{X}}_{t_k}, \bar{\mathrm{U}}_{t_k})$ given, compute $(\bar{\mathrm{X}}_{t_{k+1}}, \bar{\mathrm{U}}_{t_{k+1}})$:

if $\bar{\mathrm{X}}_{t_k} + \Delta t \bar{\mathrm{U}}_{t_k} \geq 0$ then
$$\begin{aligned} \bar{\mathrm{X}}_{t_{k+1}} &= \bar{\mathrm{X}}_{t_k} + \Delta t \bar{\mathrm{U}}_{t_k} \\ \bar{\mathrm{U}}_{t_{k+1}} &= \bar{\mathrm{U}}_{t_k} + \Delta t b(\bar{\mathrm{U}}_{t_k}) + (W_{t_{k+1}} - W_{t_k}). \end{aligned}$$

else
$$\begin{aligned} \tau_k &= t_k + \bar{\mathrm{X}}_{t_k} / \bar{\mathrm{U}}_{t_k}. \\ \bar{\mathrm{X}}_{t_{k+1}} &= -(t_{k+1} - \tau_k)\bar{\mathrm{U}}_{t_k} \\ \bar{\mathrm{U}}_{t_{k+1}} &= \underbrace{-\bar{\mathrm{U}}_{t_k} - (\tau_k - t_k) b(\bar{\mathrm{U}}_{t_k})}_{-\bar{\mathrm{U}}_{\tau_k}} + (t_{k+1} - \tau_k) b(-\bar{\mathrm{U}}_{t_k}) + (W_{t_{k+1}} - W_{t_k}). \end{aligned}$$

Weak convergence of the Euler scheme

Lemma

If $b(u) = -cu$ then $h(t, x, u)$ have bounded spatial derivatives up to the order 4 and

$$\left|\mathbb{E}f(\mathrm{X}_T) - \mathbb{E}f(\bar{\mathrm{X}}_T)\right| \leq C\Delta t$$

for f in $C_b(\mathbb{R})$.

Where

$h(t, x, u) = \mathbb{E}\left(f(\mathrm{X}_T^{t,x,u})\right)$ solves the following PDE in $[0, T] \times \mathbb{R}^+ \times \mathbb{R}$:

$$\begin{cases} \dfrac{\partial h}{\partial t} + u\nabla_x h + b(u)\nabla_u h + \dfrac{1}{2}\Delta_u h = 0, \\ h(t, 0, u) = h(t, 0, -u), \\ h(T, x, u) = f(x). \end{cases}$$

Incompressible stochastic Lagrangian model in the torus

Work in progress with J. Fontbona and J-F, Jabir

Goal : well-posedness for the following process in $\mathbb{T}^d \times \mathbb{R}^d$:

$$
\begin{aligned}
\mathrm{X}_t &= [\mathrm{X}_0 + \int_0^t \mathrm{U}_s ds] \, mod \, \mathbb{1} \\
\mathrm{U}_t &= \mathrm{U}_0 + \int_0^t \left[\langle U_s \rangle - \mathrm{U}_s - \frac{\nabla \mathscr{P}(s, \mathrm{X}_s)}{\rho(s, \mathrm{X}_s)} \mathbb{1}_{\{\rho(s, \mathrm{X}_s) > 0\}} \right] ds + \sigma W_t.
\end{aligned}
\tag{2}
$$

with

$$
\mathcal{L}aw(\mathrm{X}_t) = \rho(t, x) dx, \quad \langle \mathrm{U}_t \rangle = \mathbb{E}(\mathrm{U}_t | \mathrm{X}_t) \tag{3}
$$

and

$$
-\Delta_x \mathscr{P}(t, x) = \sum_{i,j=1}^{d} \partial^2_{ij} \left(\mathbb{E}(\mathrm{U}^i_t \mathrm{U}^j_t | \mathrm{X}_t = x) \rho(t, x) \right), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^d. \tag{4}
$$

Incompressible stochastic Lagrangian model in the torus

Work in progress with J. Fontbona and J-F, Jabir

Lemma

Assume that the nonlinear problem (2),(3),(4) *has a solution* (X,U) *such that*

$$\mathbb{E}|\mathrm{U}_t|^2 < \infty \,\forall t \in [0,T], \quad \mathbb{E}\int_0^T |\mathrm{U}_s|^2 ds < \infty \text{ and } \int_0^T \int_{\mathbb{T}^d} |\nabla \mathscr{P}(s,x)| dxds$$

Moreover, assume that for all $1-$*periodic function* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *of class* C^1 *we have* $\int_{\mathbb{R}^{2d}} \nabla(\varphi(x)\cdot u)\tilde{\rho}(0,x,u)dxdu = 0$. *Then,*

a) $\mathbb{E}(\nabla\varphi(\mathrm{X}_t)\cdot\mathrm{U}_t) = 0$ *for all* $t \in [0,T]$, *and*

b) *the process* $(\mathrm{X}_t, t \in [0,T])$ *is stationary.*

Existence result for the Vlasov Fokker Planck :

Following Carlen & Gangbo 03, via constrained steepest descent in the 2-Wasserstein metric (work in progress).

Conclusion
Applications in Computational Fluid Dynamics

Let $\mathcal{D}$ be an open set of $\mathbb{R}^3$, and a velocity V_{ext} given at $\partial\mathcal{D}$:

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t = & \Big[-\frac{1}{\rho} \nabla \langle \mathscr{P} \rangle (t, \mathrm{X}_t) \\ & \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, \mathrm{X}_t)}{k(t, \mathrm{X}_t)} \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t)\right) \Big] dt \\ & + \sqrt{C_0 \varepsilon(t, \mathrm{X}_t)} dW_t \\ & + \sum_{0 < s \leq t} 2 \left(V_{\text{ext}}(s, \mathrm{X}_s) - \mathrm{U}_{s^-} \right) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}}. \end{cases}$$

The jump term should ensure that

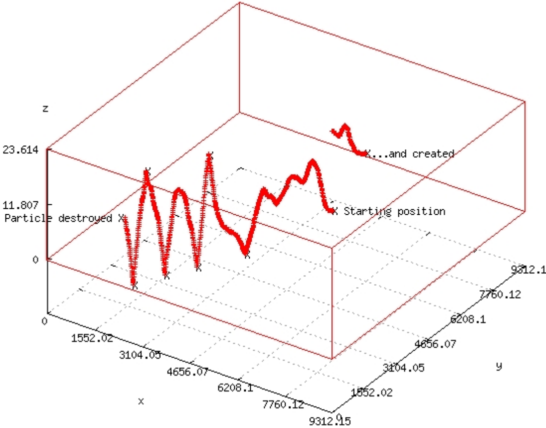
$$\langle \mathscr{U} \rangle (t, x) = V_{\text{ext}}(t, x), \quad \text{for all } x \in \partial\mathcal{D}.$$

The Stochastic Downscaling Method

A stochastic particle method to refine locally the computation scale.

- Calibrate the coefficients C_ε, ℓ_m on a simple model.
  - No relief.
- Test made on $6 \times 6 \times 6$ cells, with $T = 25\,h$.

  - $V^{(1)}_{\text{MM5}} \sim -1 m/s$,
    $V^{(2)}_{\text{MM5}} \sim -8 m/s$,
    $V^{(3)}_{\text{MM5}} \sim 0.0005 m/s$.
  - $\Delta t = 1 s$
  - Run $\sim 8\,h$ for $N_{pc} = 800$.
  - Standard deviation σ independent of N_{pc}.
  - Small spin-up.



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