Regularization properties for the 2D Bolzmann equation

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Modena, september 2010

1.The 2D Bolzmann equation

The mechanism

$$V_{k+1} = V_k + \frac{1}{2}(R_\theta - I)(V_k - v_k)$$
$$= \frac{V_k + v_k}{2} + R_\theta \times \frac{V_k - v_k}{2}$$

with

$$V_k \sim f_k(dv) \quad v_k \sim f_k(dv)$$
$$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \quad \text{with "intensity"} \quad |V_k - v_k|^\gamma \times \frac{1}{|\theta|^{1+\nu}}$$

The analytical equation

$$\partial_t f_t(v) = \int_{R^2} dv_* \int_{\pi/2}^{\pi/2} \Big(f_t(v') f_t(v'_*) - f_t(v) f_t(v_*) \Big) \frac{|v - v_*|^\gamma}{|\theta|^{1+\nu}} d\theta$$

with

$$v' = \frac{v + v_*}{2} + R_\theta \times \frac{v - v_*}{2}, \quad v'_* = \frac{v + v_*}{2} - R_\theta \times \frac{v - v_*}{2}.$$

The probabilistic interpretation (Tanaka,1978)

$$V_t = V_0 + \int_0^t \int \frac{1}{2}(R_\theta - I)(V_{s-} - v)1_{\{u \le |V_{s-}-v|^\gamma\}} N(ds, du, d\theta, dv)$$

with N Poisson point measure of compensator

$$\widehat{N}(ds, du, d\theta, dv) = dsduf_s(dv)\frac{d\theta}{|\theta|^{1+\nu}}$$

where $f_t(dv)$ is **the solution of the analytical equation**. Then

$$P(V_t \in dv) = f_t(dv).$$

The apporximaiting eqaution : we take

$$\widehat{N}_{\varepsilon}(ds, du, d\theta, dv) = dsduf_t(dv)\frac{d\theta}{\varepsilon \vee |\theta|^{1+\nu}}$$

Construction : we take

$$\lambda_\varepsilon = \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\varepsilon \vee |\theta|^{1+\nu}} < \infty,$$

$$\mu_\varepsilon(d\theta) = \frac{1}{\lambda_\varepsilon} \times \frac{d\theta}{\varepsilon \vee |\theta|^{1+\nu}} \quad \text{probability law}$$

and

$$T_{k+1} - T_k \sim 1_{[0,\infty)}(t)\frac{1}{\lambda_\varepsilon}\exp(-\lambda_\varepsilon t)$$

$$\theta_k \sim \mu_\varepsilon(d\theta) \quad v_k \sim f_{T_k}(dv) \quad U_k \sim 1_{[0,1]}(u)du$$

and the aproximating equation is

$$V^\varepsilon_{T_{k+1}} = V^\varepsilon_{T_k} + \frac{1}{2}(R_{\theta_k} - I)(V^\varepsilon_{T_k} - v)1_{\{U_k \le \left|V^\varepsilon_{T_k} - v_k\right|^\gamma\}}$$

Theorem

$$E\left|V_t - V^\varepsilon_t\right| \le C\varepsilon^{1+\gamma}$$

Alternative representation

$$\overline{V}^{\varepsilon}_{T_{k+1}} = \overline{V}^{\varepsilon}_{T_k} + \frac{1}{2}(R_{\overline{\theta}_k} - I)(\overline{V}^{\varepsilon}_{T_k} - v)$$

with

$$\overline{\theta}_k \sim \left|V^{\varepsilon}_{T_k} - v_k\right|^{\gamma} \times \mu_{\varepsilon}(d\theta).$$

Conclusion : $\overline{V}^{\varepsilon}_t$ is a functional of :

$$\begin{aligned} T_k &\sim \mathrm{exp}(\lambda_{\varepsilon}) \quad \text{colision times} \\ v_k &\sim f_{T_k}(dv) \quad \text{colision directions} \\ \overline{\theta}_k &\sim \left|V^{\varepsilon}_{T_k} - v_k\right|^{\gamma} \times \mu_{\varepsilon}(d\theta) \quad \text{colision angels} \end{aligned}$$

Global Law of the colision angels

$$(\overline{\theta}_1, ..., \overline{\theta}_n) \sim \prod_{k=1}^{n} \left|V^{\varepsilon}_{T_k} - v_k\right|^{\gamma} \times \mu_{\varepsilon}(d\theta_k)$$

Main result.

Suppose that

$$f_0(dv) \neq \delta_a \quad \forall a \in R^2$$

1. Suppose also that

$$0 < \nu < \frac{1}{3}, \quad 1 > \gamma > \frac{2\nu(1+\nu)}{1-3\nu}$$

Then for every $t > 0$

$$f_t(dv) = \phi_t(v)dv$$

with $\phi_t \in L^2(R^2, dv)$.

2. Suppose that

$$0 < \nu < \frac{1}{4}, \quad 1 > \gamma > \frac{3\nu(2+\nu)}{1-4\nu}$$

Then ϕ_t continuous.

2 Integration by parts formulas

Definition. Let F be a rnadom variable. We say that $IP_\alpha(F)$ holds if there exists $H_\alpha(F) \in \cap_{p\geq 1} L^p$ such that

$$E(\partial_\alpha \phi(F)) = E(\phi(F) H_\alpha(F)) \quad \forall \phi \in C_c^\infty(R^d).$$

Malliaivn Calculus (In a Gaussian framwork) :

For a simple functional F_n one defines the Malliavin derivative DF_n and then take the extension : if

$$F_n \to F, \quad \text{in } L^2$$

Then one defines $DF = \lim_n DF_n$. Using such a differential calculus one bilts up $IP_\alpha(F)$.

Elementary Integration by parts : Suppose that

$$F_n = f_n(\Theta_1, ..., \Theta_n)$$

and

$$(\Theta_1, ..., \Theta_n) \sim p(\theta_1, ..., \theta_n) d\theta_1, ..., d\theta_n.$$

Then, using **elementary integration by parts** one establishes

$$E(\partial_\alpha \phi(F_n)) = E(\phi(F_n) H_\alpha(F_n)) \quad \forall \phi \in C_c^\infty(R^d).$$

Regularity of the law : Suppose that $IP_\alpha(F), |\alpha| \leq d$ hold. Then $P_F(dx) = p_F(x)dx$.

Proof.

$$|E(\partial_\alpha \phi(F))| = |E(\phi(F)H_\alpha(F))| \leq \|\phi\|_\infty E(|H_\alpha(F)|)$$

and this implies

$$P_F(dy) = p_F(y)dy \quad and \quad p_F \in C_b.$$

Alternative approach : One employes **ellementary** integration by parts to bilt $IP_\alpha(F_n)$ and then

$$\begin{aligned}|E(\partial_\alpha \phi(F))| &= \lim_n |E(\partial_\alpha \phi(F_n))| = \lim_n |E(\phi(F_n)H_\alpha(F_n))| \\ &\leq \|\phi\|_\infty \sup_n E(|H_\alpha(F_n)|).\end{aligned}$$

Conclusion : We need

$$a) \quad F_n \rightarrow F \quad \text{in law},$$
$$b) \quad \sup_n E(|H_\alpha(F_n)|) < \infty.$$

In our framework

$$\sup_n E(|H_\alpha(F_n)|) = \infty.$$

Why ? We have

$$p(\overline{\theta}_1, ..., \overline{\theta}_n) = \prod_{k=1}^{n} \left| V^{\varepsilon}_{T_k} - v_k \right|^{\gamma} \times \frac{1}{\varepsilon \vee |\theta_k|^{1+\nu}}$$

$$\ln p(\overline{\theta}_1, ..., \overline{\theta}_n) = \sum_{k=1}^{n} (\gamma \ln \left| V^{\varepsilon}_{T_k} - v_k \right| - \ln \varepsilon \vee |\theta_k|^{1+\nu})$$

so

$$\partial_{\theta_i} \ln p = -\partial_{\theta_i} \ln \varepsilon \vee |\theta_i|^{1+\nu} + \sum_{k=i+1}^{n} \gamma \partial_{\theta_i} \ln \left| V^{\varepsilon}_{T_k} - v_k \right|$$

Approach using the Fourier transform. We have to prove that

$$\int_R |\xi|^p \, |\widehat{p}(\xi)| \, d\xi < \infty \quad with \quad \widehat{p}(\xi) = E(\exp(i\xi F)).$$

We denote

$$\widehat{p}_n(\xi) = E(\exp(i\xi F_n)), \quad \varepsilon_n = E\,|F - F_n|\,.$$

And we write

$$|\widehat{p}(\xi)| \le |\widehat{p}(\xi) - \widehat{p}_n(\xi)| + |\widehat{p}_n(\xi)| \le |\xi|\, \varepsilon_n + \frac{1}{|\xi|^k} E(\partial_x^k \exp(i\xi F_n))$$
$$\le |\xi|\, \varepsilon_n + \frac{1}{|\xi|^k} E(|H_k(F_n)|).$$

Toy example :

$$\varepsilon_n = \frac{1}{n} \quad and \quad E(|H_k(F_n)|) = n^2.$$

Then

$$|\widehat{p}(\xi)| \le \frac{|\xi|}{n} + \frac{n^2}{|\xi|^k} \quad \forall \xi \in N.$$

Equilibrium between n and ξ : We take

$$n = |\xi|^{\rho}$$

Then

$$|\widehat{p}(\xi)| \leq |\xi|^{1-\rho} + \frac{1}{|\xi|^{k-2\rho}} \quad \forall \xi, n \in N.$$

We want

$$1 - \rho = -(k - 2\rho) \quad \Leftrightarrow \rho = \frac{k+1}{3}$$

Conclusion : if we choose

$$n(\xi) = |\xi|^{\frac{k+1}{3}}$$

then we obtain

$$|\widehat{p}(\xi)| \leq |\xi|^{\frac{-k+2}{3}}$$

so that

$$\int_R |\xi|^p \, |\widehat{p}(\xi)| \, d\xi < \infty \quad \forall p \in N.$$

Conclusion : We write

$$|\widehat{p}_F(\xi)| \leq |\xi|\, E\,|F - F_n| + \frac{1}{|\xi|^k} E(|H_k(F_n)|)$$

and we have to estimate in an accurate way

$$\textbf{Approximation rate} : \quad E\,|F - F_n| \to 0,$$
$$\textbf{Blow up rate} \quad |H_k(F_n)| \to \infty.$$

We we are lucky we get an **equilibrium**.

Integration by parts : Toy example

We consider an one dimensional random variable

$$\Theta \sim p(\theta)d\theta$$

and a "functional"

$$F = f(\Theta).$$

Then

$$E(\phi'(F)) = E(\phi(F))$$

Proof. **Step1**.

$$\phi'(F) = \phi'(f(\Theta)) = \partial_{\Theta}\left(\phi(f(\Theta))\right) \times g(\Theta) \quad with \quad g(\Theta) = \frac{1}{f'(\Theta)}.$$

Worning : non degeneracy : $f'(\theta) \neq 0$.

Step 2.

$$
\begin{aligned}
E(\phi'(F)) &= \int \partial_\theta \left(\phi(f(\theta))\right) \times g(\theta) \times p(\theta) d\theta \\
&= -\int \phi(f(\theta) \times \left(g'(\theta) \times p(\theta) + g(\theta) \times p'(\theta)\right) d\theta \\
&= -\int \phi(f(\theta) \times \left(g'(\theta) + g(\theta) \times \frac{p'(\theta)}{p(\theta)}\right) \times p(\theta) d\theta \\
&= -E(\phi(F) \left(g'(\Theta) + g(\Theta) \times \frac{p'(\Theta)}{p(\Theta)}\right))
\end{aligned}
$$

so that

$$
H(F) = -\left(g'(\Theta) + g(\Theta) \times \partial_\Theta \ln p(\Theta)\right).
$$