

Dirichlet problems for Ornstein-Uhlenbeck operators in Hilbert spaces

July 4, 2010

We consider a family of self-adjoint Ornstein–Uhlenbeck operators \mathcal{L}_α in an infinite dimensional Hilbert space H having the same gaussian invariant measure $\mu = \mathcal{N}_Q$,

$$\mathcal{L}_\alpha\varphi(x) = \frac{1}{2} \operatorname{Tr} [Q^{1-\alpha} D^2\varphi(x)] - \frac{1}{2} \langle x, Q^{-\alpha} D\varphi(x) \rangle,$$

where $Q \in \mathcal{L}(H)$ is a symmetric positive operator with finite trace, and $0 \leq \alpha \leq 1$.

We study the Dirichlet problem for the equation $\lambda\varphi - \mathcal{L}_\alpha\varphi = f$ in a closed set $K \subset H$, with $f \in L^2(K, \mu)$. Its variational solution, easily provided by the Lax–Milgram theorem, can be represented by means of the transition semigroup stopped to K , as in finite dimensions.

We address two problems: the meaning of the Dirichlet boundary condition and the regularity of the solution φ (which belongs to a Sobolev space $W_\alpha^{1,2}(K, \mu)$ by definition) of the Dirichlet problem.

Concerning the boundary condition we consider both irregular and regular boundaries. In the first case we content to have a solution whose null extension outside K belongs to $W_\alpha^{1,2}(H, \mu)$. In the second case we exploit the Malliavin’s theory of surface integrals, to give a meaning to the trace of φ at ∂K and to show that it vanishes, as it is natural.

Concerning regularity, we can prove interior $W_\alpha^{2,2}$ regularity results. Regularity up to the boundary is much more complicated; however we have some partial results. For instance, we can treat the case $\alpha = 0$ for the unit ball and for halfspaces $K = \{x \in H : \langle b, x \rangle = 1\}$ with $b \in H$, $\|b\| = 1$.

The talk is based on joint works with G. Da Prato.